Electron interactions in graphene in a strong magnetic field

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Overview

- Recent experiments: integer QHE in graphene
- Graphene background
  - band structure; zero-energy states
- Continuum theory in the presence of a magnetic field
  - dispersion of Landau levels; length/energy scales
- Interactions and SU(4) (spin × chirality) symmetry
  - what’s special about $n = 0$
  - pseudopotentials and possible FQHE
  - easy-plane anisotropy due to ‘backscattering’ in $n \neq 0$
  - chirality ferromagnetism; effective stiffness
- Outlook
What is graphene?

- Graphene = 2D graphite
- Graphite = stack of weakly coupled graphene sheets
- Honeycomb lattice = triangular lattice with two-atom basis

Electron interactions in graphene in a strong magnetic field
What is graphene?

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- Low-tech sample preparation: Scotch tape
- Technical difficulty: good contacts
**IQHE in graphene**


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**Density of states**

Graphene IQHE:

\[ R_H = \frac{h}{e^2 \nu} \]

at \( \nu = 2(2n+1) \)

Usual IQHE:

at \( \nu = 2n \)

(no Zeeman)

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**Figures**

- **a**
  - **Density of states**
  - \( V_g = 15 \text{V} \)
  - \( T = 30 \text{mK} \)
  - \( R = \frac{1}{\nu} \)

- **b**
  - **Density of states**
  - \( B = 9 \text{T} \)
  - \( T = 1.6 \text{K} \)
  - \( \sim \nu \)

- **c**
  - **Graphene IQHE:**
  - \( R_H = \frac{h}{e^2 \nu} \)
  - at \( \nu = 2(2n+1) \)
Graphene bandstructure papers from the ’50ies

- Band-structure calculation in the tight-binding model

\[ H_0 = -t \sum_{i \in A} \sum_{j=1}^{3} \left( b_{R_i+e_j}^\dagger a_{R_i} + \text{H.c.} \right) \]

Diagram showing the lattice structure with different interaction paths.
Graphene bandstructure papers from the ’50ies

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\[ H_0 = -t \sum_{i \in A} \sum_{j=1}^{3} \left( b_{R_i}^+ e_j a_{R_i} + \text{H.c.} \right) \]

Energy dispersion:

\[ \epsilon_k = \pm t \sqrt{ \left( \sum_{j=1}^{3} \cos(k \cdot e_j) \right)^2 + \left( \sum_{j=1}^{3} \sin(k \cdot e_j) \right)^2 } \]
Zero-energy states

- Zero-energy states: $H|\psi\rangle = 0 \Rightarrow \sum_{j:i} b_{R_i+e_j} = 0$ or $\varepsilon_{K}^{\pm} = 0$
Zero-energy states

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- Can choose wavefunction amplitudes \( b_j = |b_j| \exp(i\phi_j) \) to be non-zero on \( B \) sublattice only

The page is discussing the zero-energy states in a triangular XY model. The states are characterized by global phase and chirality. The location of Dirac points at \( K \) is also mentioned.
Zero-energy states

- Zero-energy states: $H|\psi\rangle = 0 \implies \sum_{j:i} b_{R_i+e_j} = 0$ or $\varepsilon^{\pm}_K = 0$

- Can choose wavefunction amplitudes $b_j = |b_j| \exp(i\phi_j)$ to be non-zero on $B$ sublattice only

- For $|b_j|$ constant, require $\sum_{j:i} \exp(i\phi_j) = 0 \mod 2\pi$
  $\implies$ GS of triangular XY model

- These are distinguished by
  - global phase
  - chirality $= \pm \Leftrightarrow K, \ K'$
  $\implies$ Location of Dirac points
Continuum limit with no magnetic field

- Zero-energy states:

\[ \varepsilon_{K}^{\pm} = 0 \iff \sum_{j=1}^{3} \cos(K \cdot e_{j}) = \sum_{j=1}^{3} \sin(K \cdot e_{j}) = 0 \]

at \( K \) and \( K' \) points of the 1st BZ
Continuum limit with no magnetic field

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at \( K \) and \( K' \) points of the 1st BZ

- Continuum limit \( \mathbf{k} = \mathbf{K}^{\pm} + \kappa \) with \( |\kappa| \ll 1/a \):

\[ \mathcal{H}^{\pm}(\kappa) = \frac{3}{2} t a \begin{pmatrix} 0 & \kappa_1 \mp i\kappa_2 \\ \kappa_1 \pm i\kappa_2 & 0 \end{pmatrix} = \hbar v_F (\kappa_1 \sigma^1 \pm \kappa_2 \sigma^2) \]
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- Energy dispersion (two-fold degenerate, chirality \( \alpha = \pm \)):
  \[ \varepsilon_{\kappa}^{\alpha = \pm} = \pm \hbar v_F |\kappa| \]
Research activity on graphene, 2005/06

- Disorder and interaction effects (Guinea et al.)
  - poor screening due to vanishing DOS at Dirac points
  - semimetal with different ferromagnetic phases
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- Mesoscopic Physics
  - Edge (zig-zag vs. armchair) states in graphene nano-ribbons (Peres, Castro Neto, Guinea; Brey, Fertig; ...)
  - Minimal conductance $\sim e^2/h$ (Beenaker’s group, Katsnelson, ...)
  - Disorder and weak (anti-)localisation (Altshuler et al, Guinea et al., Khveshchenko, ...)

Electron interactions in graphene in a strong magnetic field

The "Castro Neto" program

"We'll have to rewrite the theory of metals for this problem."

(Physics Today, January 2006, p. 21)
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“Naive” continuum limit with magnetic field (I)

- Peierls substitution + minimal coupling to the gauge field:

\[ k \rightarrow \frac{p}{\hbar} \rightarrow \frac{1}{\hbar}(p + eA) \equiv \frac{\Pi}{\hbar} \]
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Electron interactions in graphene in a strong magnetic field – p.9/28
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- “Quantification” (magnetic length: \( l_B = \sqrt{\hbar/eB} \)):

\[ [x_\mu, p_\nu] = i\hbar \delta_{\mu,\nu} \Rightarrow [\Pi_x, \Pi_y] = -i\hbar^2/l_B^2 \]
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\]

- Ladder operators \([a, a^\dagger] = 1\) (harmonic oscillator):

\[
a = \frac{l_B}{\sqrt{2\hbar}} (\Pi_y + i \Pi_x), \quad a^\dagger = \frac{l_B}{\sqrt{2\hbar}} (\Pi_y - i \Pi_x)
\]
“Naive” continuum limit with magnetic field (II)

- Hamiltonians for the two chiralities (at $K$ and $K'$):

\[
H_K = \sqrt{2} \frac{\hbar v_F}{l_B} \begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix}, \quad H_{K'} = \sqrt{2} \frac{\hbar v_F}{l_B} \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix}
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- Energy dispersion (degenerate in chirality quantum number $\alpha$):

$$\epsilon_n = \pm \frac{\hbar v_F}{l_B} \sqrt{n} \propto \sqrt{B|n|}$$

[Relativistic Landau levels (LLs)]
Quantum states and degeneracies

- Degeneracy of relativistic LLs:
  - Symmetry due to guiding centers $\mathbf{R} = (X, Y)$:
    $[H, \mathbf{R}] = 0$, $[X, Y] = il_B^2$, states: $|n, m\rangle = |n\rangle \otimes |m\rangle$
  - “usual” orbital degeneracy: $N_\phi = A/2\pi l_B^2$
  - filling factor: $\nu = N_{el}/N_\phi$
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  - Filling factor: $\nu = N_{el}/N_\phi$
  - Chirality $\alpha = \pm$ and spin $\sigma = \uparrow, \downarrow$ (evtl. lifted by Zeeman)
    \[
    \Rightarrow \text{internal SU(4) symmetry}
    \]
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    \( \Rightarrow \) internal SU(4) symmetry

- States \( |n, m; \alpha\rangle \) are 2-spinors:
  \[
  |n, m; +\rangle = \begin{pmatrix}
  |n, m\rangle \\
  \text{sgn}(n)|n - 1, m\rangle
  \end{pmatrix},
  |n, m; -\rangle = \begin{pmatrix}
  \text{sgn}(n)|n - 1, m\rangle \\
  |n, m\rangle
  \end{pmatrix}
  \]

- Special case \( n = 0 \):
  electrons at \( K (K') \) live on \( A (B) \) sublattice only!
**Infrared transmission spectroscopy**

Grenoble high-field group: Sadowski et al., cond-mat/0605739

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Transition C

\[
\text{transmission energy [meV]} = 100 \times \sqrt{B} - 50
\]

\[
\text{relative transmission}
\]

---

Transition B

\[
\text{transmission energy [meV]} = 50 \times \sqrt{B} - 20
\]

\[
\text{relative transmission}
\]

---

Electron interactions in graphene in a strong magnetic field
“Correct” continuum limit – Harper equation

- Define wavefunctions $g_{\{A/B\}}$ for sublattices $A/B$

Near $K$ (in units of $a \equiv 1$):

$$E_{g_A}(x) = -2 \cos \left\{ \frac{2\pi}{3} + \frac{\sqrt{3}}{2} \left[ q_y + (x + 1/4)B \right] \right\} g_B(x + 1/2) - g_B(x - 1)$$

$$E_{g_B}(x) = -2 \cos \left\{ \frac{2\pi}{3} + \frac{\sqrt{3}}{2} \left[ q_y + (x - 1/4)B \right] \right\} g_A(x - 1/2) - g_A(x + 1)$$

- In Landau gauge $A = Bxe_y$, $q_y$ is good quantum number
“Correct” continuum limit – Harper equation

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  Near $K$ (in units of $a \equiv 1$):
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  \]

- In Landau gauge $A = B x e_y$, $q_y$ is good quantum number

- Problem for expansion: $B x$ is unbounded

$\Rightarrow$ For given $q_y \in [0, 2\pi]$, define auxiliary q.n. $m$ such that
  \[
  \sqrt{3}/2(q_y + B x_m) = 2\pi m \quad \text{and write} \quad x = x_m + \delta x
  \]
“Correct” continuum limit – Harper equation

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\[
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\]

and write \( x = x_m + \delta x \)

• Strategy

  - solve for given \( m, q_y \) assuming \( \delta x \) small
  - Check consistency of solution with assumption
From Harper to Dirac

- Expansion near $K, K'$ gives Dirac equation:

$$
E g_{\alpha}(x) = -\frac{3}{2} \left( \frac{d}{dx} \pm B \delta x \right) g_{\beta}(x)
$$

$$
E g_{\beta}(x) = -\frac{3}{2} \left( -\frac{d}{dx} \pm B \delta x \right) g_{\alpha}(x)
$$
From Harper to Dirac

- Expansion near $K, K'$ gives Dirac equation:
  \[
  E g_\alpha(x) = -\frac{3}{2}(d/dx \pm B\delta x)g_\beta(x) \\
  E g_\beta(x) = -\frac{3}{2}(-d/dx \pm B\delta x)g_\alpha(x)
  \]

- Tunnelling between solutions $m \neq m'$ suppressed?
  \[\Rightarrow\] Require: $R_L = \sqrt{n l_B} \sim \sqrt{n/B} \ll 1/B \sim \Delta x_n$

Electron interactions in graphene in a strong magnetic field – p.14/28
From Harper to Dirac

• Expansion near $K, K'$ gives Dirac equation:
  \[ Eg_\alpha(x) = -3/2(d/dx \pm B\delta x)g_\beta(x) \]
  \[ Eg_\beta(x) = -3/2(-d/dx \pm B\delta x)g_\alpha(x) \]

• Tunnelling between solutions $m \neq m'$ suppressed?

⇒ Require: $R_L = \sqrt{nl_B} \sim \sqrt{n/B} \ll 1/B \sim \Delta x_n$

• Equivalent to $E \ll t$ and $\rho \ll 1/a^2 \sim 10^{15} \text{cm}^{-2}$
**Length and energy scales in graphene**

**Length scales**
- Distance between neighbouring carbon atoms: $a = 0.14\text{nm}$
- Magnetic length: $l_B = 26\text{nm}/\sqrt{B[T]}$
- Larmor radius: $R_L = \sqrt{n}l_B$

---

**Energy scales**
- Band width: $t = 2.7\text{eV}$
- Lattice effects (anisotropies, etc.): $a = l_B = 0.005\text{nm}/\sqrt{B[T]}$
- Landau level `spacing': $\sim v_F = l_B = 3\tau = 20\text{meV}$
- Landau level dispersion (tunneling): $\exp(-RL = a)$
- Zeeman splitting: $\Delta = g\mu_B B = 0.1\text{meV}$
- Interaction energy: $e^2 = l_B = 2.4\cdots 12\text{meV}$
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Length and energy scales in graphene

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- Landau level dispersion (tunneling): \( \exp(-R_L/a) \)
- Zeeman splitting: \( \Delta_z = g\mu_B B \sim 0.1B \text{[T]} \text{meV} \)
- Interaction energy: \( e^2/\epsilon l_B \sim 2.4...12 \sqrt{B \text{[T]}} \text{meV} \)
**Interaction model – densities**

- Electrons in a single relativistic LL at $\nu \neq n$ (no spin):

$$H = \frac{1}{2} \sum_{q} V(q) \rho_{n}^{n}(-q) \rho_{n}^{n}(q), \quad V(q) = \frac{2\pi e^{2}}{\epsilon q}$$

$$\rho_{n}^{n}(q) = \rho_{A}^{n}(q) + \rho_{B}^{n}(q)$$

$$\rho_{\tau}^{n}(r) = \sum_{\alpha, \alpha'} \psi_{n;\alpha;\tau}^{\dagger}(r) \psi_{n;\alpha';\tau}(r)$$

with $\tau = A/B$, $\alpha = \pm$ (chirality)
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\]

\[
\rho^n(\mathbf{q}) = \rho^n_A(\mathbf{q}) + \rho^n_B(\mathbf{q})
\]

\[
\rho^n_{\tau}(\mathbf{r}) = \sum_{\alpha, \alpha'} \psi_{n;\alpha;\tau}^\dagger(\mathbf{r}) \psi_{n;\alpha';\tau}(\mathbf{r})
\]

with \( \tau = A/B \), \( \alpha = \pm \) (chirality)

• Projected densities \( \rho^n(\mathbf{q}) = \sum_{\alpha, \alpha'} F_{n}^{\alpha\alpha'}(\mathbf{q}) \bar{\rho}^{\alpha\alpha'}(\mathbf{q}) \):

\[
\bar{\rho}^{\alpha\alpha'}(\mathbf{q}) = \sum_{m, m'} \langle m | e^{-i[\mathbf{q}+(\alpha-\alpha')\mathbf{K}] \cdot \mathbf{R}} | m' \rangle c_{n, m, \alpha}^\dagger c_{n, m', \alpha'}
\]
Graphene form factors $l_B \equiv 1$:

\[
F_{n}^{++}(q) = \frac{1}{2} \left[ L_{|n|} \left( \frac{|q|^2}{2} \right) + L_{|n|-1} \left( \frac{|q|^2}{2} \right) \right] e^{-|q|^2/4} = F_{n}^{--}(q) \equiv F_{n}(q)
\]

\[
F_{n}^{+-}(q) = \left( \frac{-i(q + q^* - K - K^*)}{2\sqrt{2|n|}} \right) L_{|n|-1}^{1} \left( \frac{|q - K|^2}{2} \right) e^{-|q-K|^2/4}
\]

\[
F_{n}^{-+}(q) = [F_{n}^{+-}(-q)]^*
\]
Interaction model (II)

- Graphene form factors $l_B \equiv 1$:

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$$

$$
F_{n}^{-+}(q) = \left[ F_{n}^{+-}(-q) \right]^* 
$$

- Model: $H = \frac{1}{2} \sum_{\alpha_1, \ldots, \alpha_4} \sum_q \psi_{n}^{\alpha_1, \ldots, \alpha_4}(q) \bar{\rho}^{\alpha_1 \alpha_3}(-q) \bar{\rho}^{\alpha_2 \alpha_4}(q)$

with interaction vertex:

$$
\psi_{n}^{\alpha_1, \ldots, \alpha_4}(q) = \frac{2\pi e^2}{\epsilon |q|} F_{n}^{\alpha_1 \alpha_3}(-q) F_{n}^{\alpha_2 \alpha_4}(q),
$$
Interaction model (II)

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F_{n}^{++}(q) = \frac{1}{2} \left[ L_{|n|} \left( \frac{|q|^2}{2} \right) + L_{|n|-1} \left( \frac{|q|^2}{2} \right) \right] e^{-|q|^2/4} = F_{n}^{--}(q) \equiv \mathcal{F}_n(q)
\]

\[
F_{n}^{+-}(q) = \left( \frac{-i(q + q^* - K - K^*)}{2\sqrt{2|n|}} \right) L_{|n|-1}^1 \left( \frac{|q - K|^2}{2} \right) e^{-|q-K|^2/4}
\]

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\]

- Model: \( H = \frac{1}{2} \sum_{\alpha_1, \ldots, \alpha_4} \sum_{q} v_n^{\alpha_1, \ldots, \alpha_4}(q) \bar{\rho}^{\alpha_1 \alpha_3}(-q) \bar{\rho}^{\alpha_2 \alpha_4}(q) \)

with interaction vertex:

\[
v_n^{\alpha_1, \ldots, \alpha_4}(q) = \frac{2\pi e^2}{\epsilon |q|} F_{n}^{\alpha_1 \alpha_3}(q) F_{n}^{\alpha_2 \alpha_4}(q),
\]

\( \Rightarrow \) No SU(2) chirality symmetry so far !!
Scattering Processes

- Terms of the form
  \[ F_{n,\alpha}(\pm q)F_{n'}^{-\alpha}(\pm q) : \]
  \[ \text{exp. suppressed} \sim \exp(-|K|/8) \]
Scattering Processes

- Terms of the form
  \[ F_n^{\alpha,\alpha}(\mp q)F_n^{\alpha',-\alpha'}(\pm q) : \]
  exp. suppressed \( \sim \exp(-|K|/8) \)

- Umklapp terms
  \[ F_n^{\alpha,-\alpha}(-q)F_n^{\alpha,-\alpha}(q) : \]
  exp. suppressed \( \sim \exp(-|K|/2) \)
Scattering Processes

- Terms of the form \( F_{n}^{\alpha,\alpha}(\mp q)F_{n}^{\alpha',-\alpha'}(\pm q) \):
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- Umklapp terms
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  \( F_{n}^{\alpha,-\alpha}(-q)F_{n}^{-\alpha,\alpha}(q) \):
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\[\Rightarrow H_{SU(2)}^{n} = \frac{1}{2} \sum_{\alpha,\alpha'} \sum_{\mathbf{q}} \frac{2\pi e^{2}}{\epsilon|\mathbf{q}|} [\mathcal{F}_{n}(\mathbf{q})]^{2} \rho^{\alpha,\alpha}(-\mathbf{q})\rho^{\alpha',\alpha'}(\mathbf{q}) + \mathcal{O}(a/l_{B})\]
SU(2) Model

- SU(2) Interaction Hamiltonian:

\[ H_{SU(2)}^n = \frac{1}{2} \sum_q v_n^G(q) \bar{\rho}(-q) \bar{\rho}(q) \]

with total projected density \( \bar{\rho}(q) = \bar{\rho}^{++}(q) + \bar{\rho}^{--}(q) \)

and effective interaction potential for graphene:

\[ v_n^G(q) = \frac{\pi e^2}{\epsilon q} e^{-q^2/2} \left[ L_{|n|} \left( \frac{q^2}{2} \right) + L_{|n|-1} \left( \frac{q^2}{2} \right) \right]^2, \quad v_0^G(q) = \frac{2\pi e^2}{\epsilon q} e^{-q^2/2} \]
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- Non-relativistic limit \( n \gg 1 \):

\[ \mathcal{F}_n(q) \approx \frac{1}{2} \left[ J_0(q\sqrt{2n} - 1) + J_0(q\sqrt{2n} + 1) \right] \approx J_0(q\sqrt{2n}) \approx L_n \left( \frac{q^2}{2} \right) e^{-q^2/4} + O(1/n) \]
Effective SU(2) interaction potentials – FQHE

The effective interaction potential in the non-relativistic and relativistic cases for different values of $n$. The behavior of the potential is most stable for $n=1$ without non-relativistic effects.

- $n=0$: Relativistic and non-relativistic
- $n=1$: Relativistic
- $n=5$: Non-relativistic

Largest difference between relativistic and non-relativistic cases is for $n=1$. Similar behavior for $n=0$ and $n=1$: "chirality polarized" FQHE states are most stable in the absence of non-relativistic effects.
• Largest difference between rel. and non-rel. case in \( n = 1 \)
• Similar behaviour of rel. interaction in \( n = 0 \) and \( n = 1 \):
Effective SU(2) interaction potentials – FQHE

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• Similar behaviour of rel. interaction in $n = 0$ and $n = 1$:
  - “chirality polarised” FQHE states most stable in $n = 1$
  - absence of non-rel. $n = 1$ physics: Pfaffian at $\nu = 5/2$?
**SU(2) symmetry-breaking terms of \( \mathcal{O}(\alpha/l_B) \)**

- Backscattering terms in \( n \neq 0 \):

\[
H_{bs} = \frac{1}{2} \sum_{\alpha} \sum_{\mathbf{q}} v_{n}^{\alpha,-\alpha}(\mathbf{q}) \bar{\rho}_{\alpha,-\alpha}(-\mathbf{q}) \rho_{-\alpha,\alpha}(\mathbf{q})
\]

with interaction \( v_{n}^{+-}(\mathbf{q}) = v_{n}^{-+}(-\mathbf{q}) \)

\[
v_{n}^{+-}(\mathbf{q}) = \frac{\pi e^2 \text{Re}(\mathbf{q} - \mathbf{K})^2}{\epsilon |\mathbf{q}|} \left[ L_{|n|-1}^1 \left( \frac{|\mathbf{q} - \mathbf{K}|^2}{2} \right) e^{-|\mathbf{q} - \mathbf{K}|^2/4} \right]^2
\]

peaked at \( \mathbf{q} = \pm \mathbf{K} \): \( v_{n}^{+-}(q) \sim e^2/\epsilon |\mathbf{K}| l_B^2 \sim (e^2/\epsilon l_B)(\alpha/l_B) \)
SU(2) symmetry-breaking terms of $\mathcal{O}(\alpha/l_B)$

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peaked at $\mathbf{q} = \pm \mathbf{K}$: $v_{n}^{+-}(\mathbf{q}) \sim e^2/\epsilon |\mathbf{K}| l_B^2 \sim (e^2/\epsilon l_B)(\alpha/l_B)$

- Electrostatics in $n = 0$:
  charge distributed homogeneously on both sublattices
  $\Rightarrow$ easy-plane anisotropy $SU(2) \rightarrow U(1)$
Chirality quantum ferromagnetism

- Exchange-driven spin ferromagnetism at $\nu = 1$ in GaAs
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- SU(2) vs. O(3) description [Review: Moon et al., PRB 51, 5138 (1995)]

$$|\Psi\rangle = \prod_m \left( \sin \frac{\theta_m}{2} e^{-i \phi_m / 2} c_{m,+}^\dagger + \cos \frac{\theta_m}{2} e^{i \phi_m / 2} c_{m,-}^\dagger \right) |0\rangle \leftrightarrow n_m = \begin{pmatrix} \sin \theta_m \cos \phi_m \\ \sin \theta_m \sin \phi_m \\ \cos \theta_m \end{pmatrix}$$
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$$\mathcal{H} = \frac{\rho_s}{2} \int d^2r [\nabla n(r)]^2 \quad \rho_s = \frac{1}{16\sqrt{2\pi}} \frac{e^2}{\epsilon l_B}$$
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- Easy-plane anisotropy due to backscattering terms in $n \neq 0$:

$$\mathcal{H}_{\text{mass}} = \Delta_z \int \frac{d^2r}{2\pi\ell_B} [n_z(r)]^2 \quad \Delta_z = \frac{3\sqrt{3}}{64\pi^3} \frac{e^2}{\ell_B^2} \frac{a}{l_B}$$
**Experimental evidence for chirality coherence**

Zhang et al. PRL 96, 236806 (2006)

- IQHE in graphene:
  \[ \sigma_{xy} = \frac{2e^2}{h}(2n + 1) \]
  at \( \nu = 2(2n + 1) \)

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Outlook: From SU(2) to SU(4) in a fixed LL (I)

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- Generators of SU(2) symmetry: projected spin densities

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\bar{S}^\mu(q) = \bar{\rho}(q) \otimes S^\mu = \frac{1}{2} \sum_{m,m;\sigma,\sigma'} \langle m | e^{-i\mathbf{q} \cdot \mathbf{R}} | m' \rangle c^\dagger_{m;\sigma} \tau^\mu_{\sigma,\sigma'} c_{m';\sigma'}
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\[
[\bar{S}^\mu(q), \bar{\rho}(q')] = 2i \sin \left( \frac{q \wedge q'}{2} \right) \bar{S}^\mu(q + q')
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$$

- Additional SU(2) symmetry – SU(2)$\otimes$SU(2):

$$
\tilde{S}^\mu(q) = \bar{\rho}(q) \otimes (S^\mu \otimes 1) \quad \tilde{I}^\nu(q) = \bar{\rho}(q) \otimes (1 \otimes \tilde{I}^\nu)
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- SU(4) skyrmion physics at \( \nu = 1 \)
  [Arovas et al., PRB 59, 13147 (1999); Ezawa, PRL 82, 3512 (1999)]
Comparison with bilayer quantum Hall systems

- Layer index $\rightarrow$ isospin $\alpha = \pm$ (in addition to physical spin)
- Explicit isospin SU(2) symmetry breaking due to different intralayer and interlayer interactions:

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- Symmetry breaking $SU(2) \rightarrow U(1)$ governed by parameter $d/l_B \sim 1$ (larger than in graphene case, $a/l_B \sim 0.02$)

- At $\nu_T = \nu_+ + \nu_- = 1$: easy-plane isospin ferromagnetism or else: $U(1)$ exciton superfluid
Conclusions (I)

- Graphene: example of relativistic electrons in a condensed matter system
  - linear energy dispersion at the two inequivalent BZ corners $K$ and $K'$
  $\Rightarrow$ two-fold chirality degeneracy at low energies
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- Graphene in a magnetic field: novel type of quantum Hall effect
  - conductivity: $\sigma_{xy} = 2 \times \frac{e^2}{h} (2n + 1)$
  - relativistic Landau levels: $E_n \propto \sqrt{B|n|}$
  - degeneracies: spin+chirality in addition to orbital degeneracy
Conclusions (II)

- Electron interactions in graphene in the QH regime:
  - effective SU(2) interaction model [with spin: SU(4)]
  - symmetry-breaking terms of order $a/l_B \sim 0.02$
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• Chirality ferromagnetism at $\nu = \pm 1$
  - responsible for observed quantum Hall effect ?
  - symmetry breaking $\text{SU}(2) \rightarrow \text{U}(1)$ in comparison with bilayer case
  - other experimental signatures ?