Quantum transport and Aharonov-Bohm effect in diffusive networks

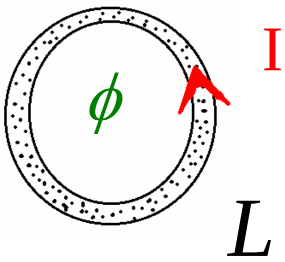
C. Texier, G. Montambaux and P. Delplace (CNRS, Université Paris-Sud, Orsay)

Persistent current in metallic mesoscopic rings

$$\mathcal{M} = IS$$
$$\phi = BS$$

$$\mathcal{M} = -\frac{\partial E_T}{\partial B}$$

$$I = -\frac{\partial E_T}{\partial \phi}$$



Phase coherence $L < L_\varphi$ — L_φ phase coherence length

Physics of disorder and interactions $l_e < L$ — Diffusive regime

Aharonov-Bohm flux dependence of mesoscopic quantities ?

JOSEPHSON BEHAVIOR IN SMALL NORMAL ONE-DIMENSIONAL RINGS

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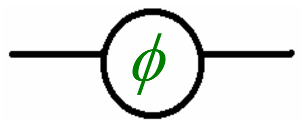
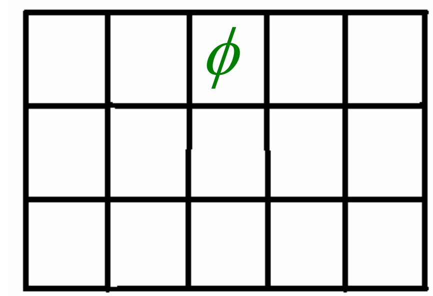
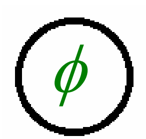
Network of diffusive quasi-1D wires





Evolution of flux dependent quantities with geometry?



Persistent current

same physics

but different structure of the diffusive trajectories

Correction to Landau diamagnetism

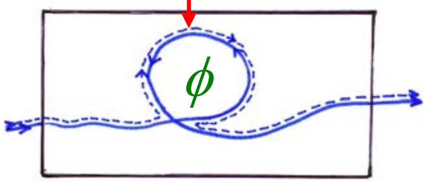
→Solve diffusion equation on a network

→Relation to persistent current and other mesoscopic quantities

A global description of mesoscopic quantities in a diffusive network

Relation to the return probability

Time reversed trajectories



Weak localization $$\Delta\sigma_\varphi = -2\int P_\varphi(t)\, e^{-\gamma t}\, dt$$

Universal conductance fluctuations $$\delta\sigma^2 \propto \int t\, P(t)\, e^{-\gamma t}\, dt$$

Persistent current $$I(\phi) \propto -\frac{\partial}{\partial\phi}\int \frac{P_\varphi(t)}{t^2}\, e^{-\gamma t}\, dt$$

$$\gamma = \frac{\hbar}{\tau_\varphi} = \frac{\hbar D}{L_\varphi^2}$$

All quantities encoded in the Laplace transform $$P_\varphi(\gamma) = \int P_\varphi(t)\, e^{-\gamma t}\, dt$$

or its harmonics

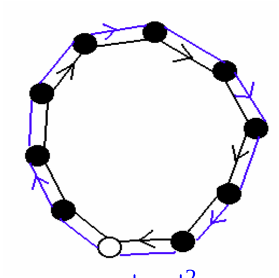
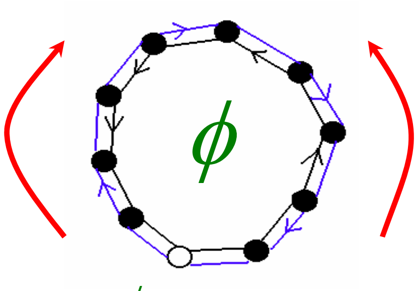
$$P_n(\gamma) = \int P_n(t)\, e^{-\gamma t}\, dt$$

What is $P_\varphi(t)$??

$$\boxed{\Delta\sigma_n(\gamma) = -2P_n(\gamma)}$$

E. Akkermans, G. Montambaux, Mesoscopic physics of electrons and photons, Cambridge, 2007

The return probability has two contributions

	$P_{cl}(t)$	$P_{\varphi}(t)$	
classical	 $\sum_j \lvert A_j \rvert^2$	$2\pi\frac{\phi}{\phi_0}$  $-2\pi\frac{\phi}{\phi_0}$ $\sum_j \lvert A_j \rvert^2 \cos 4\pi\frac{\phi}{\phi_0}$ « cooperon » $\frac{\phi_0}{2} = \frac{h}{2e}$	interference

The cooperon $P_{\varphi}(t)$ is solution of a covariant diffusion equation :

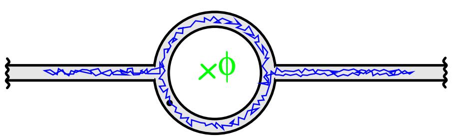
$$\left[\gamma - D\left(\nabla + i\,2e\frac{\vec{A}}{\hbar}\right)^2\right] P_{\varphi}(\gamma) = \delta(r - r')$$

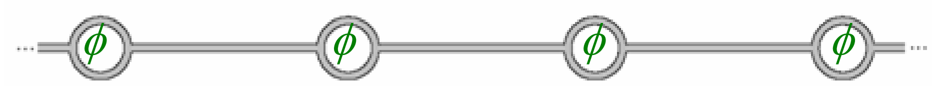
Simple example : the ring

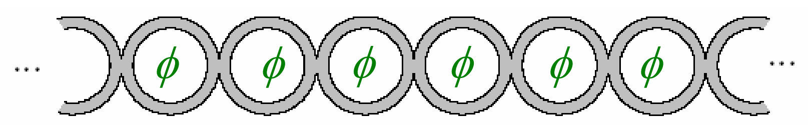
$n_t \propto \sqrt{t}$

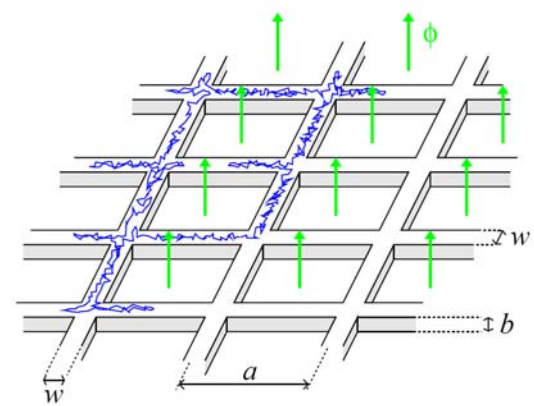
$$P_n(t) \propto \frac{e^{-\frac{n^2 L^2}{4Dt}}}{\sqrt{4\pi Dt}} \longrightarrow \boxed{\Delta\sigma_n \sim L_\varphi e^{-nL/L_\varphi}}$$

Altshuler, Aronov, Spivak, 1981









A systematic study ? on any network ?

$P_n(t)$ $\Delta\sigma_n$

Physical quantity $\propto \int t^\alpha P(t)\ e^{-\gamma t}\ dt$

Solve diffusion equation on networks

The spectral determinant

$$D\Delta P = \frac{\partial}{\partial t} P$$

$$\downarrow$$

$$D\,\Delta P = -E_i P$$

$$\int t^{\alpha} P(t) e^{-\gamma t} dt$$

$$P(t) = \sum_i e^{-E_i\, t}$$

(free space : $E_i = Dq^2$)

$$P(\gamma) = \int P(t) e^{-\gamma t} dt = \sum_i \frac{1}{\gamma + E_i} = \frac{\partial}{\partial \gamma} \ln \prod_i (\gamma + E_i) = \frac{\partial}{\partial \gamma} \ln S(\gamma)$$

All informations about diffusion
are contained in the spectral determinant

$$S(\gamma) = \prod_i (\gamma + E_i)$$

$$\int t^{\alpha} P(t) e^{-\gamma t} dt = \left(\frac{\partial}{\partial \gamma} \right)^{\alpha + 1} \ln S(\gamma)$$

Spectral determinant on a network

Solve diffusion equation on each bond
Current conservation at the nodes

$$S(\gamma) = \left(\frac{L_\varphi}{L}\right)^{N_B - N} \prod_{\alpha\beta} \sinh \frac{L_{\alpha\beta}}{L_\varphi} \det M$$

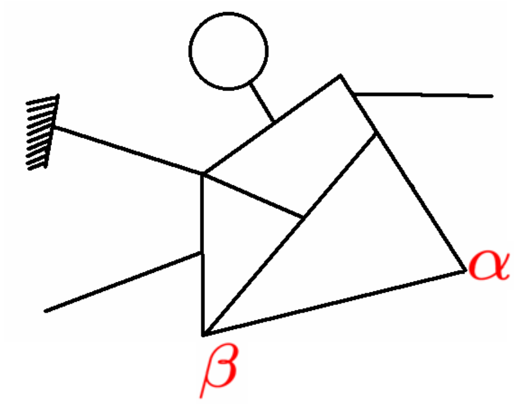
N nodes

N_B bonds

$N \times N$ matrix with elements

$$M_{\alpha\alpha} = \sum_\beta \coth \frac{L_{\alpha\beta}}{L_\varphi}$$

$$M_{\alpha\beta} = -\frac{e^{i\theta_{\alpha\beta}}}{\sinh \frac{L_{\alpha\beta}}{L_\varphi}}$$



$$\theta_{\alpha\beta} = \frac{4\pi}{\phi_0} \int_\alpha^\beta \vec{A}.\vec{dl}$$

$$P(\gamma) = \frac{\partial}{\partial\gamma} \ln S(\gamma)$$

$1/L_\varphi$

$\downarrow$

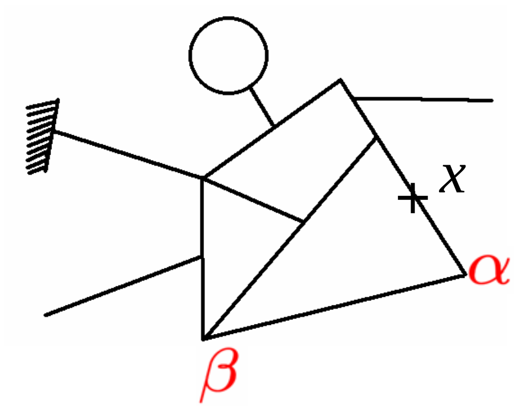
cf: superconducting networks (Alexander, De Gennes) → GL equation i/ξ

scattering on 1D graphs (C. Texier, G.M.) → Schrödinger eq. ik

*M. Pascaud, G.M., PRL **82**, 4512 (1999)*

What about return probability in one point ?

$$S(\gamma) = \left(\frac{L_\varphi}{L}\right)^{N_B - N} \prod_{\alpha\beta} \sinh\frac{L_{\alpha\beta}}{L_\varphi} \det M$$



Introduce

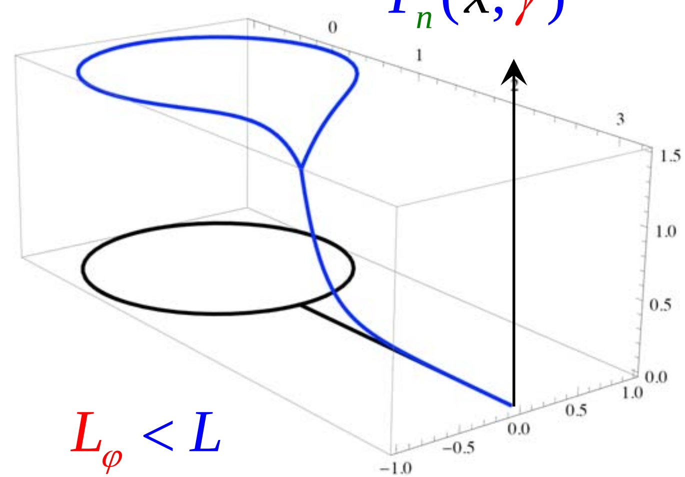
$$M_{\alpha\alpha} = \sum_\beta \coth\frac{L_{\alpha\beta}}{L_\varphi} + \lambda_x$$

$$M_{\alpha\beta} = -\frac{e^{i\theta_{\alpha\beta}}}{\sinh\frac{L_{\alpha\beta}}{L_\varphi}}$$

$$\theta_{\alpha\beta} = \frac{4\pi}{\phi_0}\int_\alpha^\beta \vec{A}.\vec{dl}$$

$$P(x,\gamma) = \frac{\partial}{\partial\lambda_x} \ln S(\lambda_x,\gamma)\Big|_{\lambda_x=0}$$

$P_n(x,\gamma)$



$L_\varphi < L$

C. Texier, G.M., J.Phys.**38**, 3455 (2005)

From spectral determinant to winding properties

$$P(\gamma) = \int P(t)\, e^{-\gamma t} dt = \frac{\partial}{\partial \gamma} \ln S(\gamma)$$

→ $P_\varphi(t)$, $P_n(t)$ and n_t

$P_\varphi(\gamma)$ — **Inv. Laplace** → $P_\varphi(t)$

Fourier ↓ $P_n(\gamma)$ — **Inv. Laplace** → $P_n(t)$ ↓ **Fourier**

Physical quantities

$\Delta\sigma_n(L_\varphi)$

$I_n(L_\varphi)$

...

Winding properties

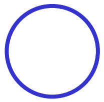
Example: the first harmonics $P_1(\gamma)$ for connected rings

Limit $L_\varphi \ll L$

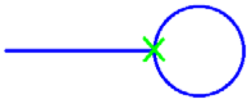
From the expansion of ln det M

Reduction by a factor $r = \prod_\alpha \left(\frac{2}{z_\alpha}\right)$

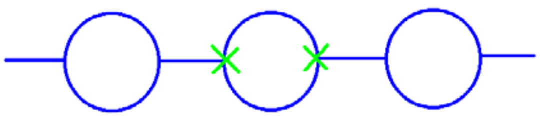
z_α coordination number



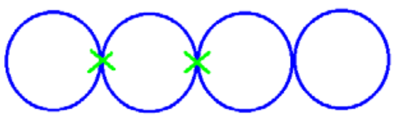
$$1$$



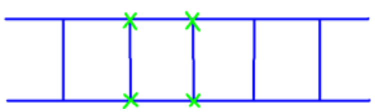
$$\frac{2}{3}$$



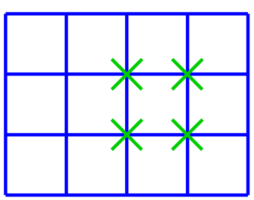
$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}$$



$$\left(\frac{2}{4}\right)^2 = \frac{1}{4}$$



$$\left(\frac{2}{3}\right)^4 = \frac{16}{81}$$



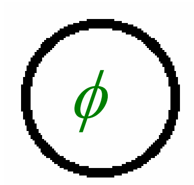
$$\left(\frac{2}{4}\right)^4 = \frac{1}{16}$$

→ There is a persistent current for connected rings

*M. Pascaud, G. M., PRL **82**, 4512 (1999)*

*W. Rabaud, L. Saminadayar et al., persistent currents in mesoscopic connected rings, **86**,3124,2001*

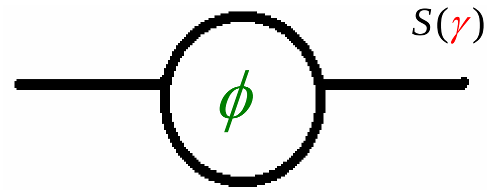
Examples



$$S(\gamma) = 2(\cosh L/L_\varphi - \cos 4\pi\varphi) \qquad \varphi = \frac{\phi}{\phi_0} \qquad \gamma = D/L_\varphi^2$$

$$\Delta\sigma \sim -\frac{L_\varphi \sinh L/L_\varphi}{\cosh L/L_\varphi - \cos 4\pi\varphi} \quad \Longrightarrow \quad \boxed{\Delta\sigma_n \sim L_\varphi e^{-n\,L/L_\varphi}}$$

AAS



$$S(\gamma) = \sinh L/L_\varphi + 2(\cosh L/L_\varphi - \cos 4\pi\varphi) = 2(\cosh L_{eff}/L_\varphi - \cos 4\pi\varphi)$$

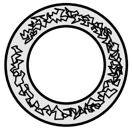
same form as for an isolated ring with L_{eff}

$$L \gg L_\varphi \qquad L_{eff} \sim L + L_\varphi \ln\frac{3}{2} \quad \Longrightarrow \quad \boxed{\Delta\sigma_n \sim \left(\frac{2}{3}\right)^n L_\varphi e^{-n\,L/L_\varphi}}$$

$$L \ll L_\varphi \qquad L_{eff} \sim \sqrt{2L\,L_\varphi} \quad \Longrightarrow \quad \boxed{\Delta\sigma_n \sim \sqrt{L\,L_\varphi}\, e^{-n\sqrt{2L/L_\varphi}}}$$

*C. Texier, G.M., J Phys A **38**, 3455 (2005)*

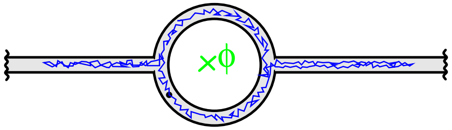
Winding properties



$$P_n(\gamma) \sim L_\varphi\, e^{-n L/L_\varphi}$$

$$P_n(t) = \frac{e^{-\frac{n^2 L^2}{4Dt}}}{\sqrt{4\pi D t}}$$

$$n_t \sim t^{1/2}$$



$L_\varphi \ll L$

$$P_n(\gamma) \sim \left(\frac{2}{3}\right)^n L_\varphi\, e^{-n L/L_\varphi}$$

$t \ll \tau_D$

$$P_n(t) = \left(\frac{2}{3}\right)^n \frac{e^{-\frac{n^2 L^2}{4Dt}}}{\sqrt{4\pi D t}}$$

$L_\varphi \gg L$

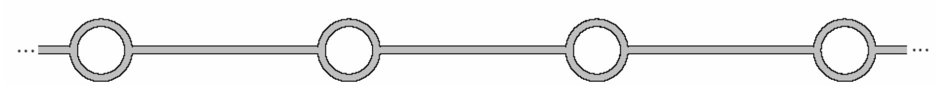
$$P_n(\gamma) \sim \sqrt{L L_\varphi}\, e^{-n\sqrt{2L/L_\varphi}}$$

$t \gg \tau_D$

$$P_n(t) \propto \frac{n^{1/3}}{t^{5/6}}\, e^{-\frac{n^{4/3}}{t^{1/3}}}$$

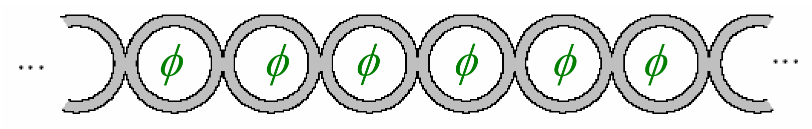
$$n_t \sim t^{1/4}$$

Anomalous diffusion



*C. Texier, G.M., J Phys A **38**, 3455 (2005)*

Winding properties



$$S_\varphi(\gamma)$$

$L_\varphi \ll L$

$$\Delta\sigma_n \sim \frac{(2n-1)!!}{2^{n+1}n!} L_\varphi e^{-nL/L_\varphi}$$

$t \ll \tau_D$

$$P_n(t) \sim -\frac{(2n-1)!!}{2^{n+1}n!}\frac{1}{\sqrt{4\pi t}} e^{-(nL)^2/4Dt}$$

$$n = 1 \;\Rightarrow\; \frac{1}{4}$$

$$n_t \sim t^{1/2}$$

$L_\varphi \gg L$

$$\Delta\sigma_n \sim -\frac{L}{2\pi} K_0\left(n\frac{L}{L_\varphi}\right)$$

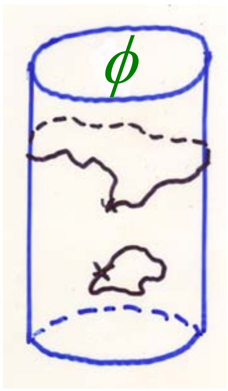
$t \gg \tau_D$

$$P_n(t) \sim \frac{L}{8\pi t} e^{-(nL)^2/4Dt}$$

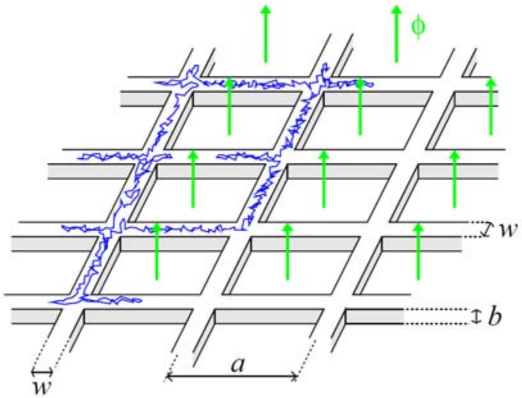
Normal diffusion

cf : diffusion on a cylinder

$$P_n(t) \sim \frac{1}{4\pi t} e^{-(nL)^2/4Dt}$$



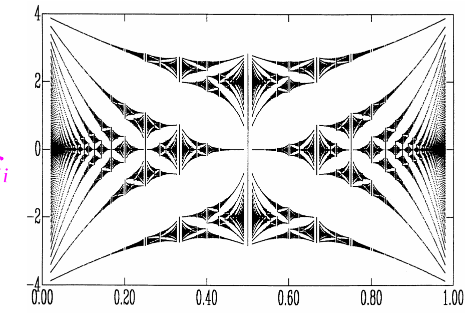
Winding properties



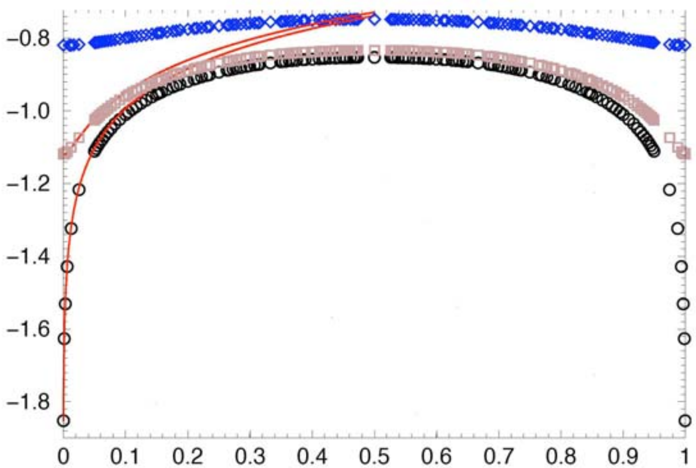
Spectral determinant = eigenvalue problem for a square lattice in a magnetic field =

Hofstadter spectrum $\varphi = \dfrac{p}{q}$

ε_i



$\varphi = p / q$



$\dfrac{L_\varphi}{a} = 2, 5, 50$

$$P_\varphi(\gamma) = \sum_i \frac{1}{\gamma + E_i} = \sum_i \frac{1}{4\cosh\sqrt{\gamma}\,a + \varepsilon_i(\varphi)}$$

$L_\varphi \gg L$

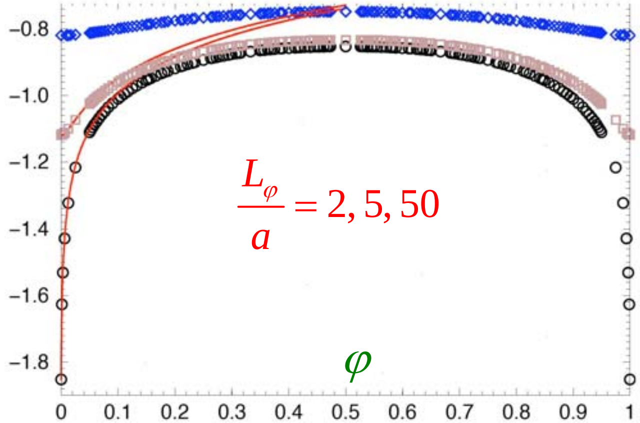
$$P_n(\gamma) \sim \frac{1}{n} f\left(\sqrt{n}\,\frac{L}{L_\varphi}\right)$$

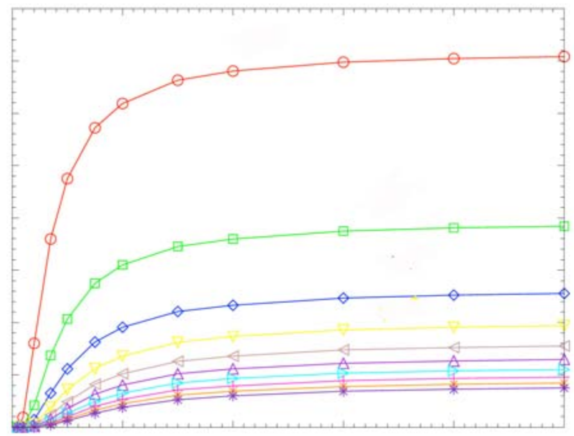
$n_t \sim t$

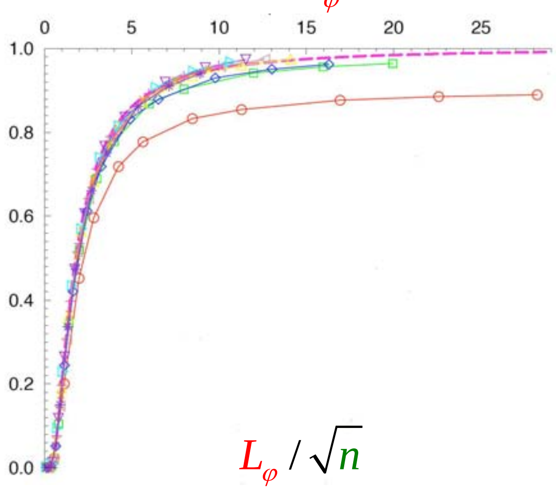
$t \gg \tau_D$

$$P_n(t) \sim \frac{a^3}{8t^2} \frac{1}{\cosh^2(\pi a^2 n / t)}$$

Levy's law







$L_\varphi \gg L$

$$P_n(\gamma) \sim \frac{1}{n} f\left(\sqrt{n}\,\frac{L}{L_\varphi}\right)$$

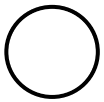
$t \gg \tau_D$

$n_t \sim t$

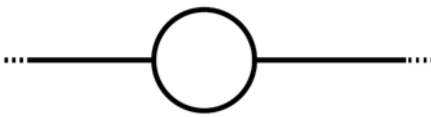
$$P_n(t) \sim \frac{a^3}{8t^2} \frac{1}{\cosh^2(\pi a^2 n / t)}$$

Levy's law

Summary

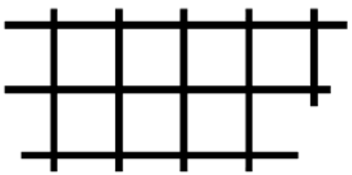


$d = 0 \qquad \alpha = 1/2$



$d = 1 \qquad \alpha = 1/4$

$d = 1 \qquad \alpha = 1/2$

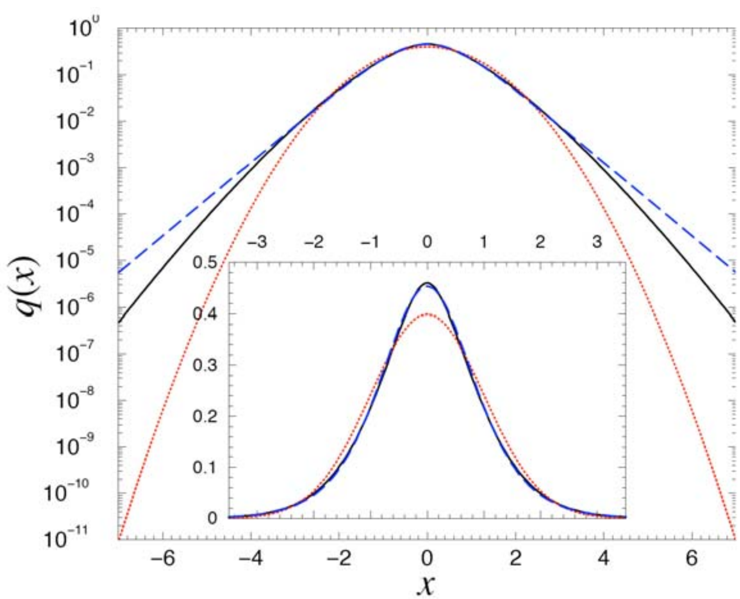


$d = 2 \qquad \alpha = 1$

$$n_t \propto t^{\alpha}$$

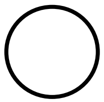
$$P_n(t) \propto \frac{1}{t^{d/2+\alpha}} q\left(\frac{n}{t^{\alpha}}\right)$$

$$\Delta\sigma_n(L_\varphi) \propto \frac{1}{L_\varphi^{d-2+2\alpha}} F_\alpha\left(\frac{n}{L_\varphi^{2\alpha}}\right)$$

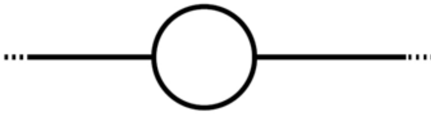


Summary

$$n_t \propto t^{\alpha}$$

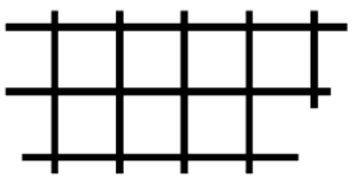


$d = 0 \qquad \alpha = 1/2$



$d = 1 \qquad \alpha = 1/4$

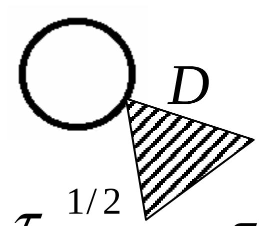
$d = 1 \qquad \alpha = 1/2$



$d = 2 \qquad \alpha = 1$

$$P_n(t) \propto \frac{1}{t^{d/2+\alpha}}\, q\!\left(\frac{n}{t^{\alpha}}\right)$$

$$\Delta\sigma_n(L_\varphi) \propto \frac{1}{L_\varphi^{d-2+2\alpha}}\, F_\alpha\!\left(\frac{n}{L_\varphi^{2\alpha}}\right)$$



$d = 0 \qquad \alpha = \dfrac{2-D}{4}$

$n_t \sim \tau_R^{1/2}$

τ_R = recurrence time = time spent near the origin

$\tau_R = t^{(2-D)/2}$

$D = 1 \rightarrow n_t \sim t^{1/4}$

$D = 2 \rightarrow n_t \sim \sqrt{\ln t}$

$D = 3 \rightarrow n_t \sim Cte$

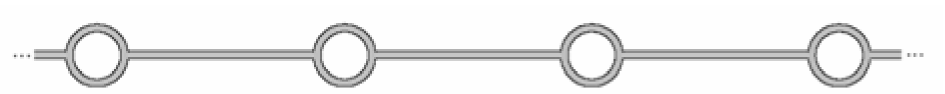
C. Texier, G.M., J Phys A **38**, 3455 (2005)

Summary

$$L_\varphi \propto T^{-1/3}$$

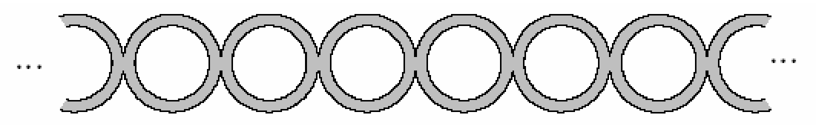
Altshuler,Aronov,Khmelnitskii

Température dependence of the weak localization correction (*Altshuler,Aronov, Spivak oscillations)*



$$\Delta\sigma_n \sim \sqrt{L L_\varphi}\, e^{-n\sqrt{2L/L_\varphi}}$$

$$\Delta\sigma_n \propto e^{-nT^{1/6}}$$

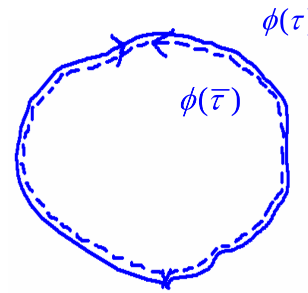


$$\Delta\sigma_n \sim \frac{L}{2\pi} K_0\left(n\frac{L}{L_\varphi}\right)$$

$$\Delta\sigma_n \propto e^{-nT^{1/3}}$$

What is the nature of decoherence ?

Weak-localization correction :



$$\Delta\sigma_n = -2\int P_n(t)\; e^{-t/\tau_\phi}\, dt$$

Diffusion

Dephasing : e-e interaction $\left\langle e^{i\Phi(t)} \right\rangle$

Altshuler,Aronov,Khmelnitskii :

e-e interaction = electric fluctuating potential → Fluctuating phase

$$\left\langle e^{i\Phi(t)} \right\rangle \sim e^{-\frac{1}{2}\left\langle \Phi^2(t) \right\rangle}$$

$$\Phi(t) = \phi(t) - \overline{\phi}(t)$$

$$\phi(t) = \frac{e}{\hbar}\int_0^t V(r(\tau),\tau)\, d\tau$$

$$\frac{d\left\langle \Phi^2(t) \right\rangle}{dt} \sim \frac{e^2}{\hbar^2}\left\langle V^2 \right\rangle_t \sim \frac{e^2}{\hbar^2} k_B T\, R_t$$

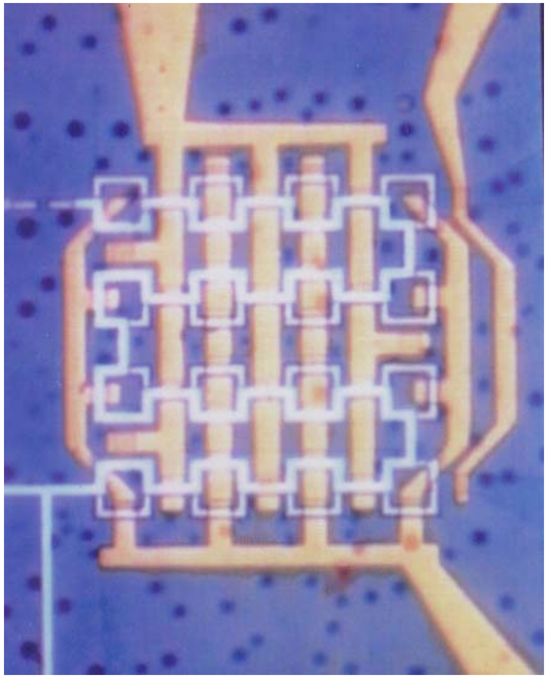
The decoherence depends on the nature of the diffusive trajectories

C. Texier, G.M.,

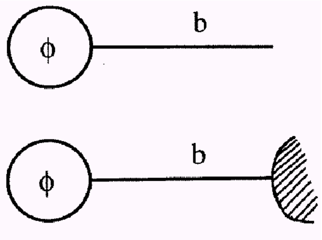
Connected rings

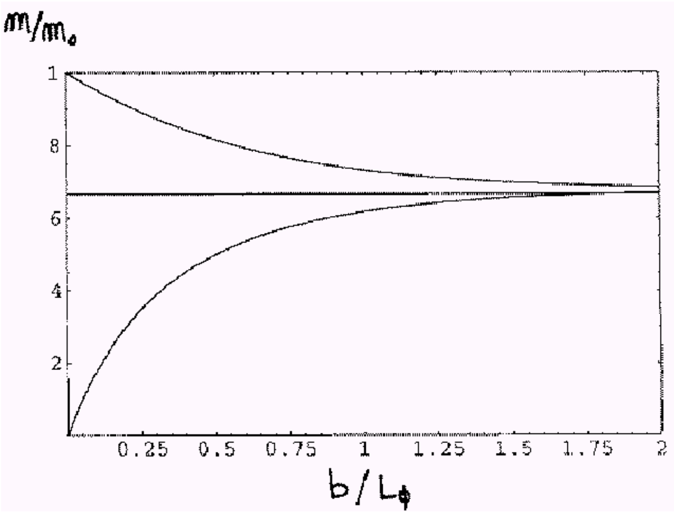
W. Rabaud, L. Saminadayar *et al.* , 2000 :

There is a finite persistent current
for connected rings.
The current is only slightly reduced.



Persistent current is not a property
of isolated rings :





$\frac{2}{3}$

Persistent current insensitive to reservoirs