# Lecture 2

**Weak localization**

- Loops and quantum crossings
- Nb loops, probability $P_{int}(r,r',t)$, return probability
- Magnetic field, phase coherence
- Weak-localisation in dimension d

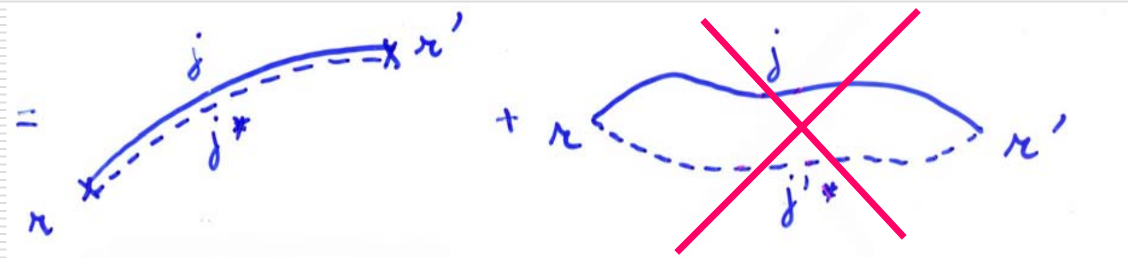
**A few solutions of the diffusion equation and WL**

- Magnetic field and negative magnetoresistance
- Magnetic field in quasi-1D wires
- AAS oscillations

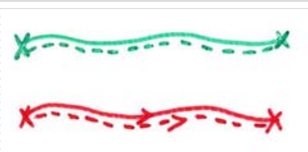
# Summary of lecture 1

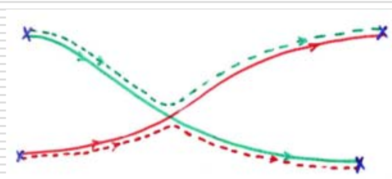
conductance ~ transmission ~ probability

$$P_{cl}(r,r')$$



classical diffuson          quantum corrections





**quantum crossing → 1/g correction**

$$p_{\times}(\tau_D) \sim \frac{\lambda_F^{d-1} v_F \tau_D}{V} \sim \frac{1}{g}$$

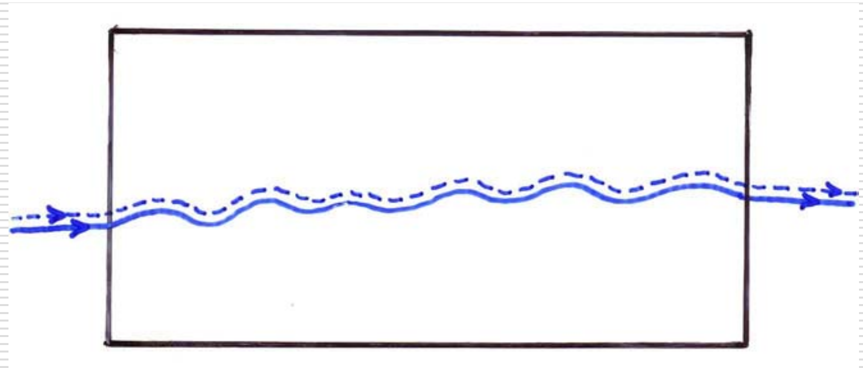
classical transport $\propto g \frac{e^2}{h}$

quantum effects $\propto \frac{e^2}{h}$

# Weak localization

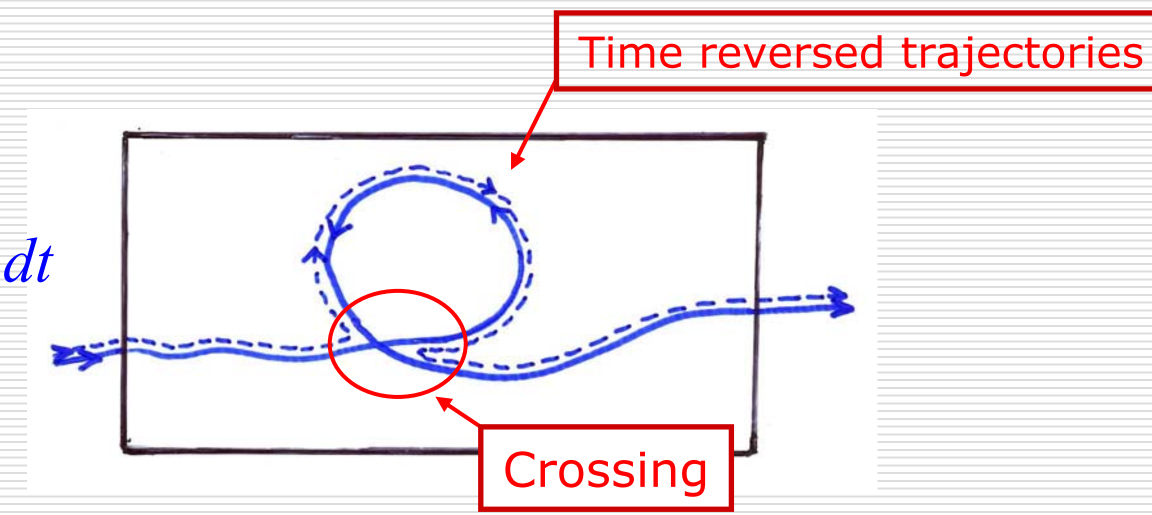
**Classical conductance**

$$G_{cl}$$



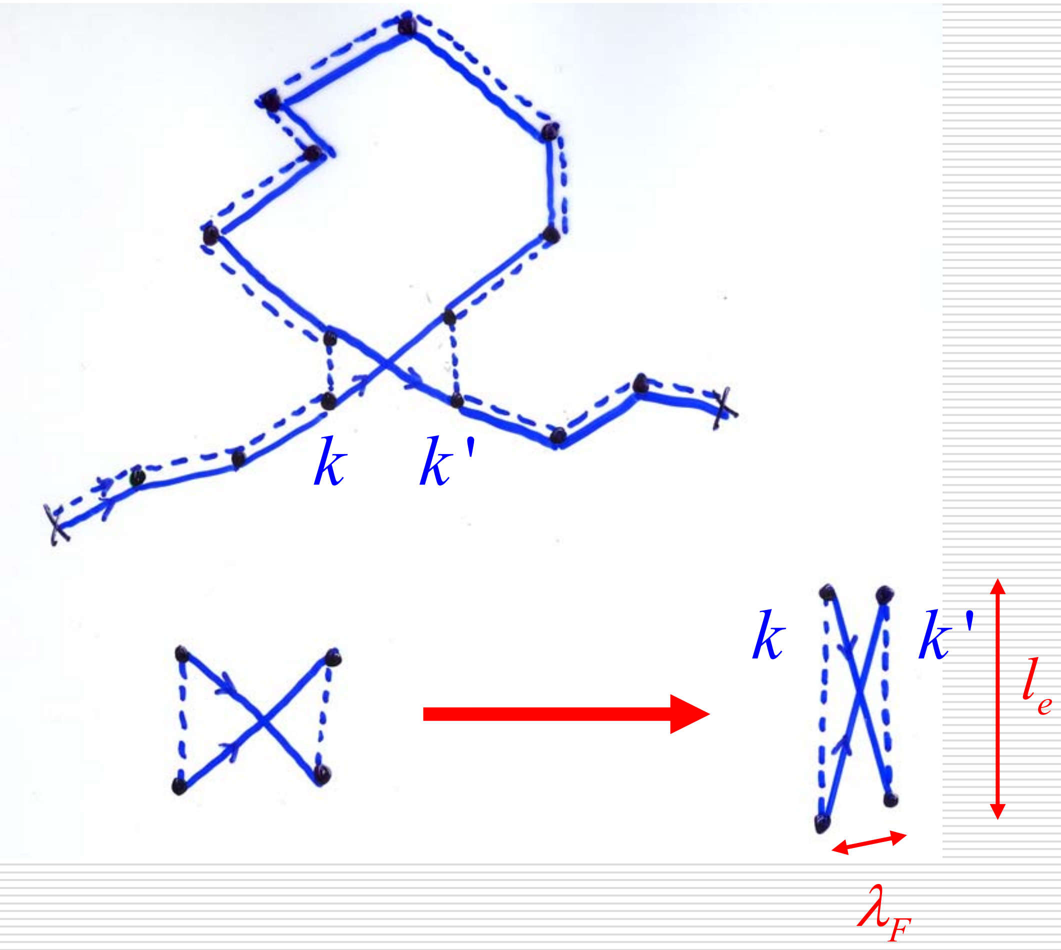
**Quantum correction** → **One crossing** → **One loop**

$$\frac{\Delta G}{G_{cl}} \sim -\frac{\lambda_F^{d-1} v_F}{V} \int_{\tau_e} P_{\text{int}}(t)\,dt$$




$P_{\text{int}}(t)$ = **distribution of number of loops with time t = return probabililty**

# (-) sign




$$k' \sim -k$$

# Weak localization

$$\frac{\Delta G}{G_{cl}} \sim -\frac{\lambda_F^{d-1} v_F}{V} \int_{\tau_e} P_{\text{int}}(t)\,dt$$

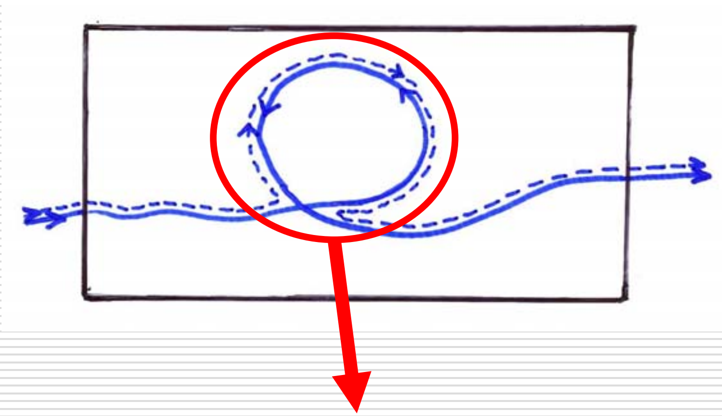
$$\downarrow$$

$$\frac{\Delta G}{G_{cl}} \sim -\frac{1}{g} \int_{\tau_e} P_{\text{int}}(t) \frac{dt}{\tau_D}$$

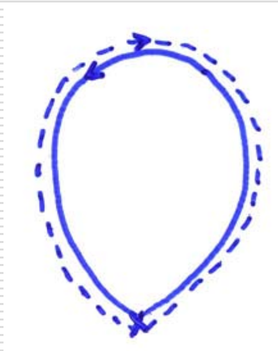
$$P_{\text{int}}(t) \qquad ?$$

## How to calculate $P_{\text{int}}(t)$ ?

$$P_{\text{int}}(t) = \int P_{\text{int}}(r,r,t)\, d^d r$$



Cooperon



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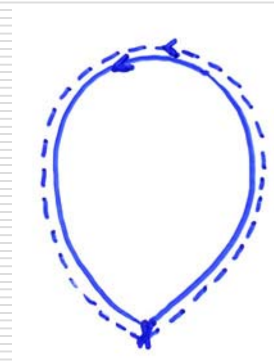


**Diffuson**

**Interference term**

$$P_{\text{int}}(r,r,t) = P_{cl}(r,r,t)$$

**Classical return probability**

**If time reversal invariance**

# Weak localization

$$\frac{\Delta G}{G_{cl}} \sim -\frac{1}{g} \int_{\tau_e} P_{\text{int}}(t) \frac{dt}{\tau_D}$$

$$P_{\text{int}}(t) = P_{cl}(t) = \frac{L^d}{(4\pi D t)^{d/2}}$$

$$= \left( \frac{\tau_D}{4\pi t} \right)^{d/2}$$

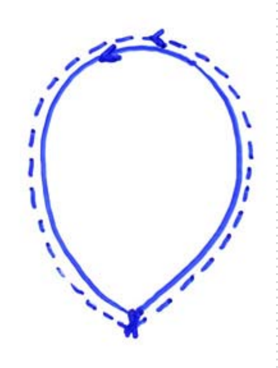
→ **The return probability *P(t)* increases for small *d***
**Coherent effets are more important in low dimension**

## Important difference :

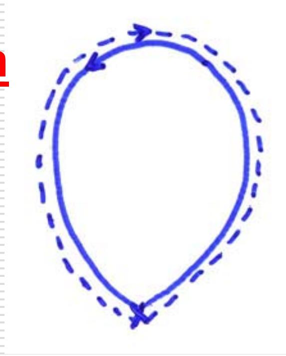
$P_{cl}(r,r',t) \Longrightarrow$ **paired trajectories follow the same direction**

$P_{\text{int}}(r,r',t) \Longrightarrow$ **paired trajectories follow opposite directions**

**Diffuson**

$A_j$  $A_j^*$

**Cooperon**

$A_j$  $A_j^{*T}$

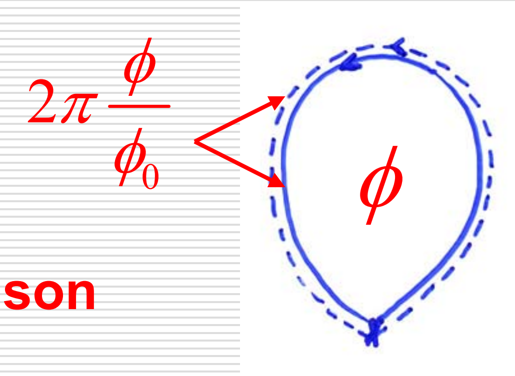
$A_j \quad A_j^T$ have the same phase $\int \vec{p}.\vec{dl}$

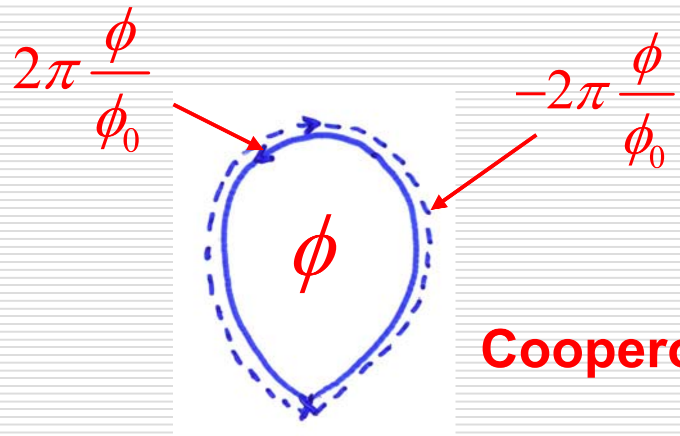
$$P_{\text{int}}(r,r,t) = P_{cl}(r,r,t)$$

**If time reversal invariance**

**If phase coherence between the reversed trajectories is preserved**

**Diffuson**




**Cooperon**

Cooperon: in a magnetic flux, paired trajectories get opposite phases

→ **phase difference** $4\pi\frac{\phi}{\phi_0}$

In a magnetic field, dephasing between time reversed trajectories
→The cooperon vanishes at large field

# Uniform magnetic field (qualitative)

$$P_{\mathrm{int}}(t) = P_{cl}(t)\left\langle e^{4i\pi\frac{\phi(t)}{\phi_0}} \right\rangle \sim e^{-t/\tau_B}$$

$$\phi(t) \sim BR^2(t) \sim BDt$$

$\phi(t) < \phi_0$

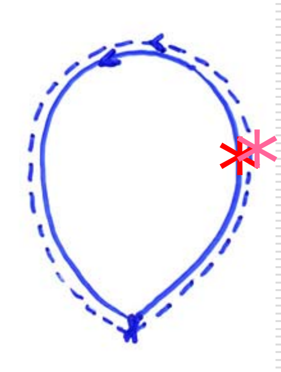
$\phi(t) > \phi_0$

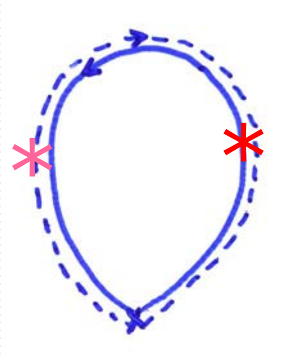
$$BD\tau_B = \phi_0$$

**Trajectories which enclose more than one flux quantum do not contribute to** $P_{\mathrm{int}}(t)$

# Phase coherence

Diffuson





Cooperon

**If some process breaks Phase Coherence, only trajectories with** $t < \tau_\phi$ **contribute**

$$P_{\text{int}}(t) = P_{cl}(t)e^{-t/\tau_\phi}\left\langle e^{4i\pi\frac{\phi(t)}{\phi_0}}\right\rangle$$

magnetic impurities
electron-phonon, electron-electron interactions

# Diffusion equation for $P_{int}(r,r,t)$ ?

$$\left[\frac{1}{\tau_\varphi}+\frac{\partial}{\partial t}-D\left(\nabla+i2e\frac{\vec{A}}{\hbar}\right)^2\right]P_{\text{int}}(r,r',t)=\delta(r-r')\delta(t)$$

Phase coherence time

Effective charge

Vector potential

$$r=r'$$

$$\left(\frac{\partial}{\partial t}-D\Delta\right)P_{cl}(r,r',t)=\delta(r-r')\delta(t)$$

# Weak localization correction

$$\frac{\Delta G}{G_{cl}} \sim -\frac{1}{g} \int_{\tau_e} P_{\text{int}}(t) \frac{dt}{\tau_D}$$

Exact result :

$$\boxed{\Delta G = -2s\frac{e^2}{h}\int_0^\infty P_{\text{int}}(t)\left(e^{-t/\tau_\phi} - e^{-t/\tau_e}\right)\frac{dt}{\tau_D}}$$

# Dependence on dimensionality

$$\Delta G = -2s\frac{e^2}{h}\int_0^\infty P_{int}(t)\left(e^{-t/\tau_\phi} - e^{-t/\tau_e}\right)\frac{dt}{\tau_D}$$

**Macroscopic system** $L \gg L_\phi \quad \tau_D \gg \tau_\phi$

$$P(t) = \left(\frac{\tau_D}{4\pi t}\right)^{d/2}$$

$$\int_{\tau_e}^{\tau_\phi} \frac{dt}{t^{d/2}}$$

$\sqrt{\tau_\phi} - \sqrt{\tau_e}$ $\quad d = 1 \quad (quasi-1D)$

$\ln\frac{\tau_\phi}{\tau_e}$ $\quad d = 2$

$\frac{1}{\sqrt{\tau_\phi}} - \frac{1}{\sqrt{\tau_e}}$ $\quad d = 3$

# Dependence on dimensionality

$$\Delta G = -s\frac{e^2}{h}\frac{L_\phi(T)}{L} \qquad d=1 \quad (quasi-1D)$$

$$\Delta G = -s\frac{e^2}{\pi h}\ln\frac{L_\phi(T)}{l_e} \qquad d=2$$

**Correction more important for small d because return probability is enhanced**

$$\Delta G = -s\frac{e^2}{2\pi h}\frac{L}{l_e} \qquad d=3$$

**Mesoscopic system** $L \ll L_\phi \qquad \tau_D \ll \tau_\phi$

$$\Delta G \sim -s\frac{e^2}{h} \qquad d=1 \quad (quasi-1D)$$

$$\Delta G \sim -s\frac{e^2}{\pi h}\ln\frac{L}{l_e} \qquad d=2$$

$$\Delta G \sim -s\frac{e^2}{2\pi h}\frac{L}{l_e} \qquad d=3$$

# Solving diffusion equation

$$P(t) = \sum_n e^{-E_n t}$$

**where** $E_n$ **are the eigenvalues of the diffusion equation** $-D\Delta\psi_n = E_n\psi_n$

**Example : uniform magnetic field in 2D**

$$E_n = \left(n + \frac{1}{2}\right)\frac{8\pi e B D}{\hbar}$$

$$P(t) = \frac{B/\phi_0}{\sinh 4\pi B D t/\phi_0}$$

**Quasi-1D wire : boundary conditions**

# Four examples

weak localization in 2 D, negative magnetoresistance

weak localization in a quasi-1d wire

weak localization in a ring

weak localization in a cylinder

# Example 1: weak localization in 2 D

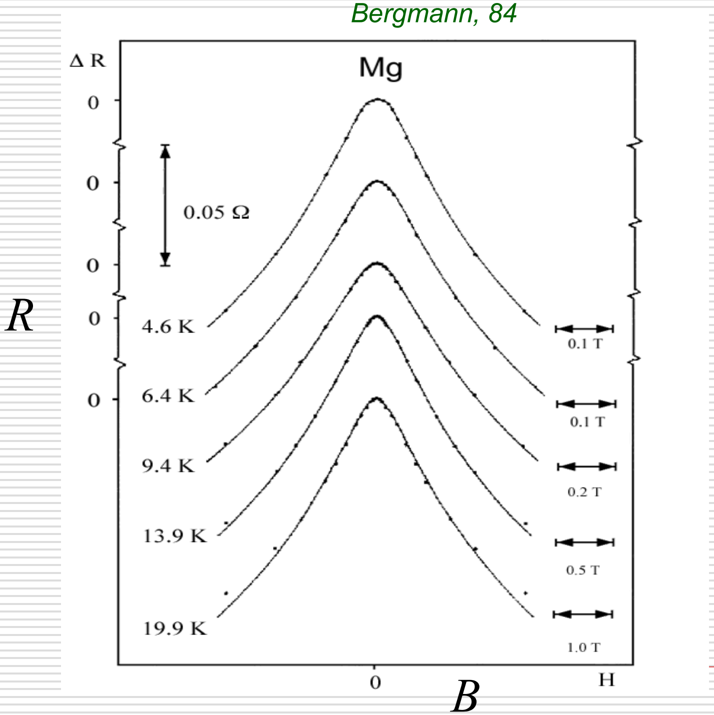
$$\frac{\Delta G}{G_{cl}} \propto -\int_{\tau_e}^{\infty} P(t)\, e^{-t/\tau_\phi}\, dt$$

$$P(t) = \frac{S}{4\pi D t} \quad \Longrightarrow \quad \Delta G \sim -\ln \frac{L_\phi}{l_e}$$

**In a magnetic field :**

$$P(t) = \frac{BS/\phi_0}{\sinh 4\pi B D t/\phi_0} \quad \Longrightarrow \quad \Delta G \sim -\ln \frac{\min\left(L_\phi, L_B\right)}{l_e} \qquad B L_B^2 \sim \phi_0$$

Bergmann, 84




**Weak localization correction is suppressed when**

$$L_B \sim L_\phi$$

$$B\, L_\phi^2 \sim \phi_0 \quad \Longrightarrow \quad L_\phi(T) \propto 1/\sqrt{T}$$

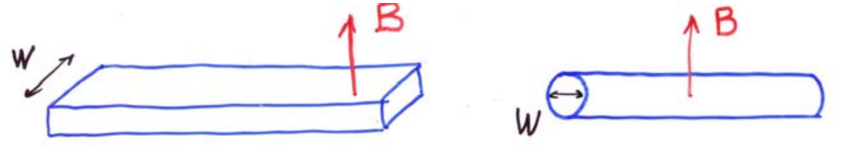
# Example 1: weak localization in 2 D

$$\Delta G = -2s\frac{e^2}{h}\int_0^\infty \frac{B/\phi_0}{\sinh 4\pi BDt/\phi_0}\left(e^{-t/\tau_\phi} - e^{-t/\tau_e}\right)\frac{dt}{\tau_D}$$

$$\Delta G = -s\frac{e^2}{2\pi h}\left[\psi\left(\frac{1}{2}+\frac{\hbar}{4eBD\tau_e}\right)-\psi\left(\frac{1}{2}+\frac{\hbar}{4eBD\tau_\phi}\right)\right]$$

# Example 2 : W.L. in a quasi-1D wire

$$\frac{\Delta G}{G_{cl}} \propto -\int_{\tau_e}^{\infty} P(t)\, e^{-t/\tau_\phi}\, dt$$




$$P_0(t) = \frac{L}{(4\pi D t)^{1/2}} \quad \Longrightarrow \quad \Delta G = -s\frac{e^2}{h}\frac{L_\phi}{L}$$

**In a magnetic field :**

$$P(t) = P_0(t)e^{-t/\tau_B} \qquad L_B^2 = D\tau_B \qquad BWL_B \sim \phi_0$$

$$\frac{1}{\tau_\phi} \rightarrow \frac{1}{\tau_\phi} + \frac{1}{\tau_B} \quad \Longrightarrow \quad \Delta G = -s\frac{e^2}{hL}\left(\frac{1}{L_\phi^2} + \frac{1}{L_B^2}\right)^{-1/2}$$

***Altshuler,Aronov***

# Example 2 : W.L. in a quasi-1D wire

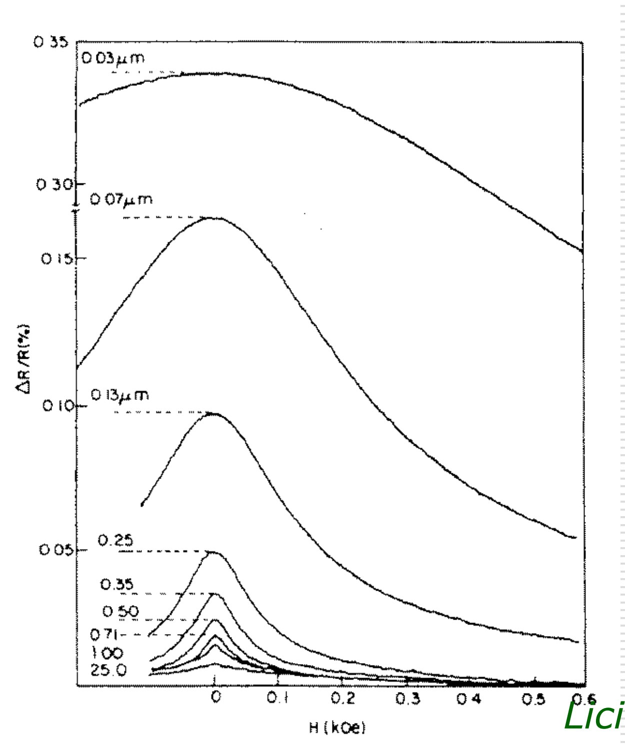
$$\Delta G = -s\frac{e^2}{hL}\left(\frac{1}{L_\phi^2}+\frac{1}{L_B^2}\right)^{-1/2}$$

$$BWL_B \sim \phi_0$$

**Weak localization correction is suppressed when**

$$L_B \sim L_\phi$$

$$BW\,L_\phi \sim \phi_0 \longrightarrow L_\phi \propto T^{-1/3}$$




*Licini,Dolan,Bishop,1980*

# Example 3 : weak localization in a ring

*Altshuler, Aronov, Spivak, '81*

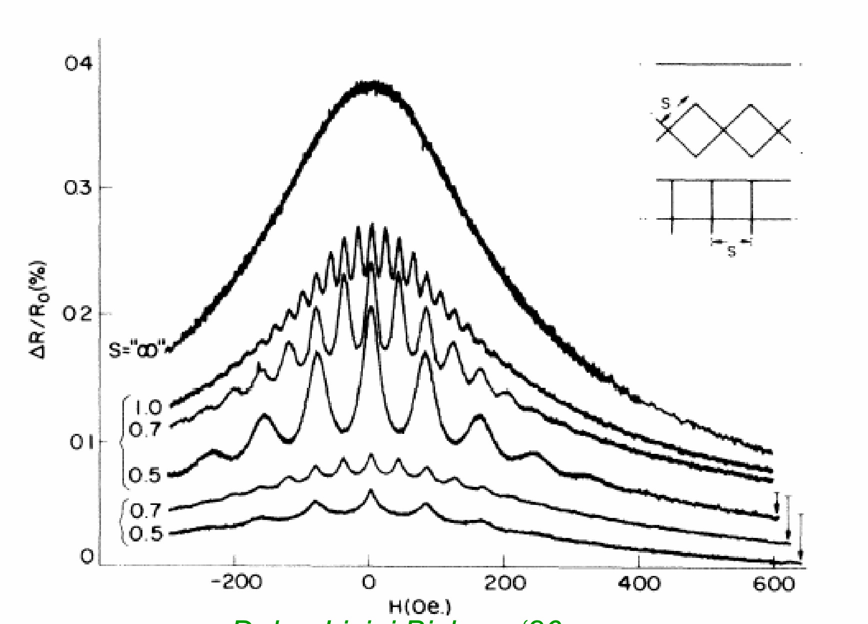
$$\frac{\Delta G}{G_{cl}} \propto -\int_{\tau_e}^{\infty} P(t)\, e^{-t/\tau_\phi}\, dt$$

**Ring in a Aharonov-Bohm flux :**

$$P(t) = \sum_m \frac{e^{-m^2 L^2/4Dt}}{\sqrt{4\pi D t}} \cos 4\pi m \frac{\phi}{\phi_0} \Longrightarrow \Delta G_m \sim -\frac{L_\phi}{L}\, e^{-m\frac{L}{L_\phi}}$$

$L_\phi$ **may be extracted either from the amplitude of the oscillations**

$$e^{-mL/L_\phi} \Longrightarrow L_\phi \propto T^{-1/3}$$



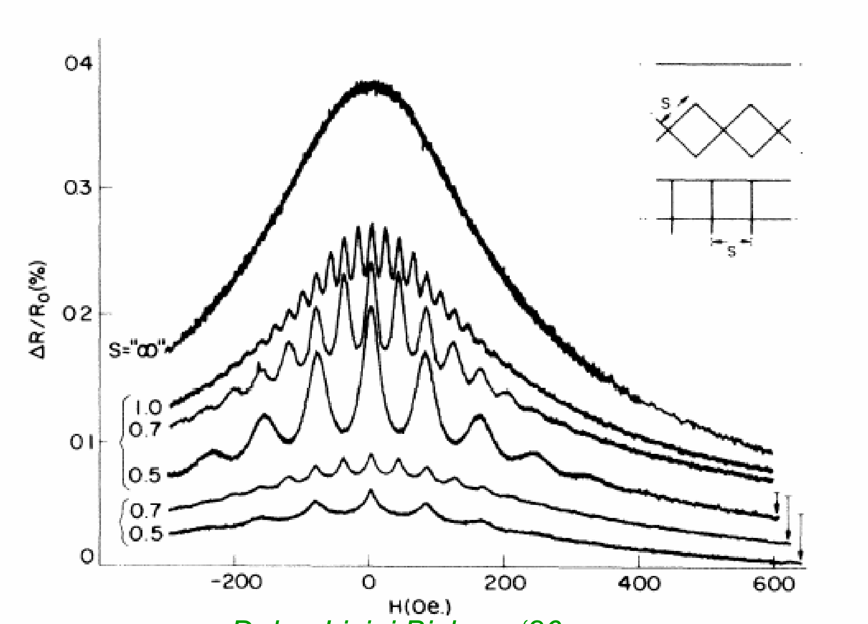

*Dolan,Licini,Bishop, '86*

# Example 3 : weak localization in a ring

penetration of the field in the ring

$$P(t)=\sum_m \frac{e^{-m^2L^2/4Dt}}{\sqrt{4\pi Dt}}\cos 4\pi m\frac{\phi}{\phi_0}\; e^{-t/\tau_B} \quad\Longrightarrow\quad \Delta G_m \sim -\frac{L_\phi}{L}\; e^{-m\frac{L}{L_\phi}}$$

$$\frac{1}{L_\phi^2} \to \frac{1}{L_\phi^2}+\frac{1}{L_B^2}$$



Dolan,Licini,Bishop, '86

$L_\phi$ may be also extracted from the envelope

$$BW\, L_\phi \sim \phi_0$$

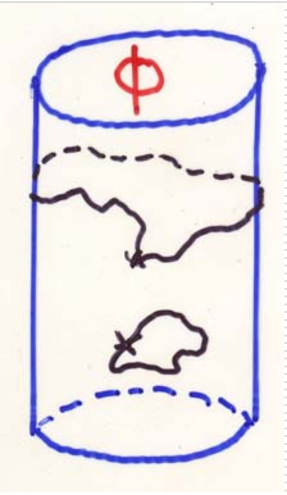
Altshuler, Aronov, Spivak, '81

## Example 4 : weak localization in a cylinder

$$\frac{\Delta G}{G_{cl}} \propto -\int_{\tau_e}^{\infty} P(t)\, e^{-t/\tau_\phi}\, dt$$

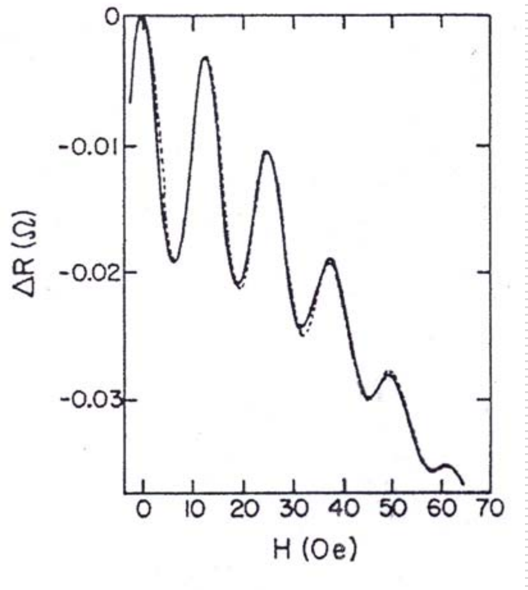
Cylinder in a Aharonov-Bohm flux :

$$P(t) = \sum_m \frac{e^{-m^2 L^2/4Dt}}{4\pi D t} \cos 4\pi m \frac{\phi}{\phi_0}\, e^{-t/\tau_B}$$




$$\Delta G_m = -s \frac{e^2}{\pi h} \left[ \ln \frac{L_\phi}{l_e} + 2 \sum_m K_0\left( m \frac{L}{L_\phi} \right) \cos 4\pi m \frac{\phi}{\phi_0} \right]$$

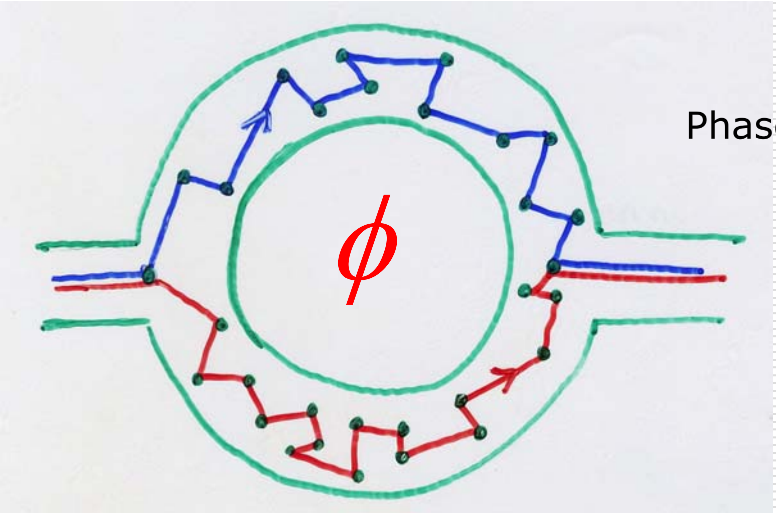
**2D diffusion** **winding of trajectories**




*Altshuler, Aronov, Spivak, '81*

*Sharvin,Sharvin, '81*

*Webb*

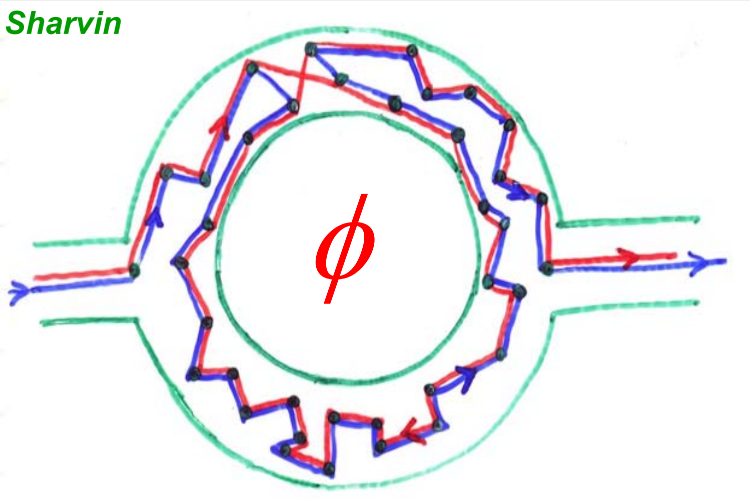



Sample specific interference

Phase difference $2\pi \frac{\phi}{\phi_0}$

Oscillates with period $h/e$

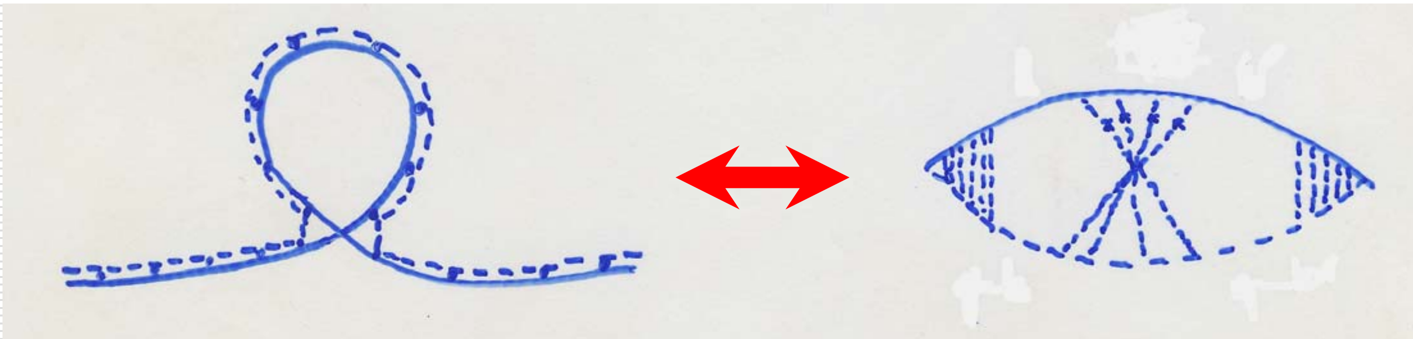
*Sharvin, Sharvin*

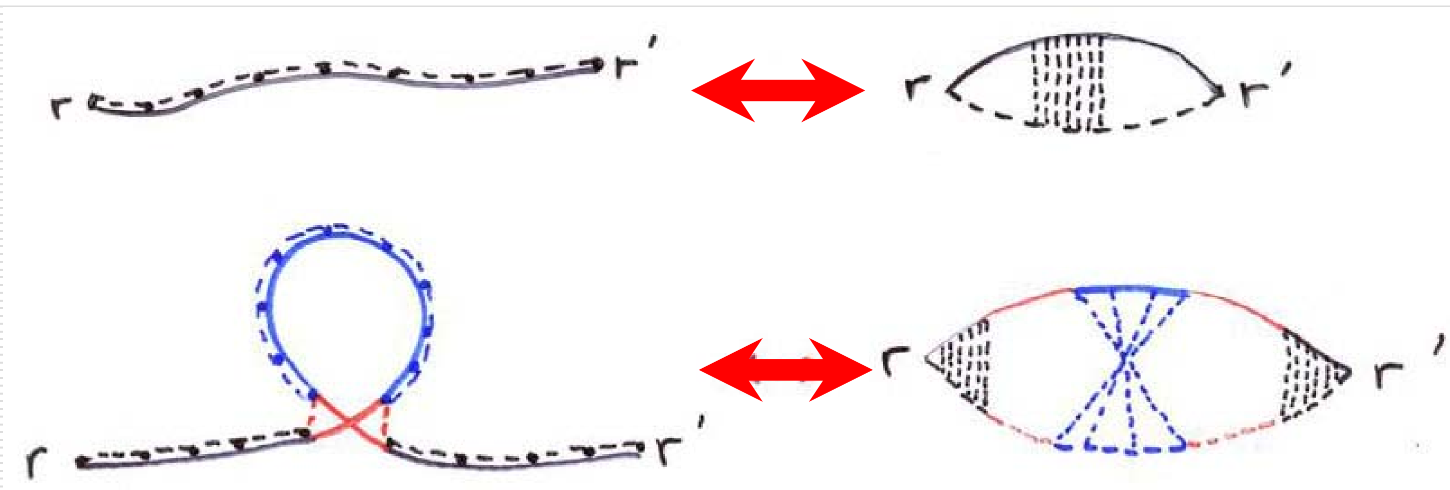



Survives disorder average

Phase difference $4\pi \frac{\phi}{\phi_0}$

Oscillates with period $h/2e$

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