Mesoscopic Physics
of Electrons and Photons

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Cover page : to be done
Wave propagation in random media has been the subject of an intense activity for more than two decades. This is now a large domain of research whose frontiers are still fuzzy and include a variety of problems such as wave localization (weak and strong), mesoscopic physics, effects of electron-electron interactions in metals, etc. Moreover, since many disorder effects are not truly specific to a given kind of waves, various approaches have been developed independently either in condensed matter physics, in optics, in atomic physics or in acoustics.

A large number of monographs or review articles already exist in the literature and they cover quite in detail various aspects of the field. Our purpose is rather to present on the one hand the basic common features of disorder effects on wave propagation and on the other hand to provide to the non-specialist reader the necessary tools needed to enter and practice this field of research.

Our first concern has been to give a description of the basic physical effects using a single formalism independent of the specific nature of the waves (electrons, electromagnetic waves, etc.). To this purpose, we have started by a detailed presentation of "single particle" average quantities such as density of states and elastic collision time using the framework of the so-called "Gaussian model" for the two most important examples of waves, namely Schrödinger and scalar Helmholtz wave equations. We have tried, as much as we could, to make precise the very basic notion of multiple scattering by an ensemble of independent effective scatterers whose scattering cross section may be obtained by means of standard one-particle scattering theory.

Nevertheless, the quantities of physical interest that are experimentally accessible to describe wave propagation in the multiple scattering regime depend essentially on the probability of quantum diffusion which describes the propagation of a wavepacket. This probability thus plays a central role and Chapter 4 is devoted to its detailed study. We see then emerging notions like classical (Diffuson) and coherent (Cooperon) contributions to the probability, which provide basic explanations to the observed physical phenomena such as weak localization corrections to electronic transport, negative magnetoresistance in a magnetic field, coherent backscattering of light as well as universal conductance fluctuations, optical speckles and mesoscopic effects in orbital magnetism.

It thus happens that all these effects result from the behavior of a single quantity, namely the probability of quantum diffusion. But in spite of this common background shared by optics and electronics of random media, each one of these domains has its own specificity which allows to develop complementary approaches. For instance, the continuous change of the relative phases of electronic wave functions that can be achieved by means of a magnetic field or a vector potential has no obvious equivalent in optics. On the other hand, it is possible in optics to change directions of incident and outgoing beams and from this angular spectroscopy to trace back correlations between angular channels.

We have made a special effort in trying to keep this book accessible to the largest audience starting at a graduate level in physics with an elementary acquaintance in quantum mechanics as a prerequisite. We have also been led to skip a number of interesting but perhaps too specialized issues among which
the study of quantum dots, relations between spectral and transport quantities, strong localization and the Anderson metal-insulator transition, electronic ballistic billiards where "quantum complexity" does not result from disorder but instead from the boundary shape and, metal-superconductor interfaces. All these aspects reflect the richness of the field of "quantum mesoscopic physics" to which this book constitutes a first introduction.

A pleasant task in finishing the writing of a book is certainly the compilation of acknowledgments to all those who have helped us at various stages of the elaboration and writing, either through discussions, criticisms and especially encouragements and support: O. Assaf, H. Bouchiat, B. Huard, J. Cayssol, C. Cohen-Tannoudji, N. Dupuis, D. Estève, A. Georges, S. Guéron, M. Kouchnir, R. Maynard, F. Piéchon, H. Pothier, B. Reulet, B. Shapiro, B. van Tiggelen, D. Ullmo, J. Vidal, E. Wolf. We wish to single out the contribution of C. Texier for his endless comments, suggestions, corrections which have certainly contributed to improve the quality of this book. Dov Levine has accepted to help us in translating the book into english. This was a real challenge and we wish to thank him for his patience in trying to teach us some good english. We also wish to thank G. Bazalitsky for carrying out most of the figures with a lot of dedication.

- Throughout this book, we use the (SI) international unit system, except in Chapter 13. The Planck constant $\hbar$ is generally taken equal to unity in particular throughout Chapter 4. In the chapters where we think that it is important to restore it, we have mentioned it at the beginning of the corresponding chapter. In order to simplify the writing, we have sometimes partially restored $\hbar$ in a given expression, especially when the correspondence between energy and frequency is straightforward.

- To keep homogeneous and consistent notations throughout a book which covers fields that are usually studied separately is a kind of challenge that, unfortunately, we have not always been able to overcome.

- We have chosen not to give an exhaustive list of references, but instead to quote papers either for their obvious pedagogical value or because they discuss a particular point presented for instance as an exercise.
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