

Persistent Currents on Networks

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We develop a method to calculate persistent currents and their spatial distribution (and transport properties) on graphs made of quasi-1D diffusive wires. They are directly related to the field derivatives of the determinant of a matrix which describes the topology of the graph. In certain limits, they are obtained by simple counting of the nodes and their connectivity. We relate the average current of a disordered graph *with interactions* and the *noninteracting* current of the same graph with clean 1D wires. A similar relation exists for orbital magnetism in general. [S0031-9007(99)09102-4]

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The existence of persistent currents in mesoscopic metallic rings is a thermodynamic signature of phase coherence [1]. These currents have been calculated using diagrammatic methods in which disorder and interactions are treated perturbatively [2–5], in a way very similar to the calculation of transport quantities such as the weak-localization (WL) correction, or the universal conductance fluctuations (UCF). Like transport quantities [6–8], they have also been derived (after disorder averaging) using semiclassical calculations, in which they were expressed in terms of the classical and interference parts of the return probability for a diffusive particle [9–12]. This formalism had made possible the calculation of WL corrections on any type of graph made of diffusive wires [13]. A diffusion equation was solved on each link of the graph with current conservation on each node. For a network with N nodes, the return probability could be related to the elements of a $N \times N$ “connectivity” matrix M and its inverse. This method has also been used recently to calculate the magnetization of such a network [14], but it required rather lengthy calculations.

In this Letter, we show that the magnetization and the transport quantities can be *directly* written in terms of the determinant $\det M$ of the connectivity matrix. Besides being a very powerful method to calculate the above quantities, this result leads to a straightforward harmonic expansion of these quantities for any network geometry. The efficiency of this method is shown for simple geometries of connected rings. In addition, we are able to derive the local distribution of the currents in the links of the network. Since the persistent current problem has still to be considered as unsolved, it is of interest to motivate new experiments in various geometries for which the magnetization and its distribution can be simply predicted and related to geometrical or topological parameters.

In the course of this work, we shall obtain a simple expression for the spectral determinant of the diffusion equation, defined as

$$S_d(\gamma) = \prod_n b_n(\gamma + E_n), \quad (1)$$

where E_n are the eigenvalues of the diffusion equation, and b_n are regularization factors [15]. Using the analogy

between the diffusion and the Schrödinger equation, we will point out a very simple relation between the Hartree-Fock (HF) average magnetization of a diffusive system and the grand canonical magnetization of the corresponding clean system. As a simple example, we relate the Aslamasov-Larkin contribution to the magnetization and the Landau susceptibility.

All quantities of interest in this work can be related to the solution $P(\vec{r}, \vec{r}', \omega)$ of the diffusion equation in a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}(\vec{r})$ [16] ($\hbar = 1$ throughout the paper):

$$[-i\omega + \gamma - D(\nabla_{\vec{r}} - 2ie\vec{A})^2]P(\vec{r}, \vec{r}', \omega) = \delta(\vec{r} - \vec{r}'). \quad (2)$$

D is the diffusion constant. Unless specified, the magnetic field dependence is implicit. $\gamma = 1/\tau_\phi = D/L_\phi^2$ is the phase coherence rate. L_ϕ and τ_ϕ are, respectively, the phase coherence length and time. In the following, we will need only the space integrated return probability $P(t) = \int d^d \vec{r} P(\vec{r}, \vec{r}, t)$. It is simply written in terms of the eigenvalues E_n of the diffusion equation, $P(t) = \sum_n e^{-(E_n + \gamma)t} = P_0(t)e^{-\gamma t}$. The time integral of $P(t)$, i.e., the Laplace transform of $P_0(t)$, can be straightforwardly written in terms of the spectral determinant (1):

$$\mathcal{P} \equiv \int_0^\infty dt P(t) = \sum_n \frac{1}{E_n + \gamma} = \frac{\partial}{\partial \gamma} \ln S_d(\gamma). \quad (3)$$

Let us now recall how average magnetizations can be written in terms of $P(t)$. Here, we restrict ourselves to $T = 0$ K. The fluctuation of the magnetization $M_{\text{typ}} \equiv (\langle M^2 \rangle - \langle M \rangle^2)^{1/2}$ is given by [11]

$$M_{\text{typ}}^2 = \frac{1}{2\pi^2} \int_0^{+\infty} dt \frac{P''(t, B) - P''(t, 0)}{t^3}, \quad (4)$$

where $P''(t, B) = \partial^2 P(t, B) / \partial B^2$. The main contribution to the average magnetization is due to electron-electron interactions [3,4]. Considering a screened interaction $U(\vec{r} - \vec{r}') = U\delta(\vec{r} - \vec{r}')$ and defining $\lambda_0 = U\rho_0$, where ρ_0 is the density of states (DoS) at the Fermi energy ϵ_F , the HF contribution to the magnetization has been written as [10]

$$\langle M_{ee} \rangle = -\frac{\lambda_0}{\pi} \frac{\partial}{\partial B} \int_0^{+\infty} dt \frac{P(t, B)}{t^2}. \quad (5)$$

Considering higher corrections in the Cooper channel leads to a ladder summation [5,12,17,18], so that λ_0 should be replaced by $\lambda(t) = \lambda_0/[1 + \lambda_0 \ln(\epsilon_F t)]$ [19]. We shall discuss later the contribution of this renormalization.

Using standard properties of Laplace transforms, the above time integrals can be written as integrals of the spectral determinant, so that the magnetizations read

$$M_{\text{typ}}^2 = \frac{1}{2\pi^2} \int_{\gamma}^{+\infty} d\gamma_1 (\gamma - \gamma_1) \frac{\partial^2}{\partial B^2} \ln S_d(\gamma_1) |0\rangle_B^B, \quad (6)$$

$$\langle M_{ee} \rangle = \frac{\lambda_0}{\pi} \int_{\gamma}^{+\infty} d\gamma_1 \frac{\partial}{\partial B} \ln S_d(\gamma_1). \quad (7)$$

In the case of a ring or a graph geometry, the integral converges at the upper limit. For the case of a magnetic field in a bulk system, this limit should be taken as $1/\tau_e$, where τ_e is the elastic time. Finally, we also recall that transport properties such as WL or UCF can be also related to the spectral determinant [14].

We now wish to emphasize an interesting correspondence between the HF magnetization of a phase coherent *interacting diffusive* system and the grand canonical magnetization M_0 of the corresponding *noninteracting clean* system. The latter can also be written in terms of a spectral determinant. The grand canonical magnetization M_0 is given quite generally by

$$M_0 = -\frac{\partial \Omega}{\partial B} = -\frac{\partial}{\partial B} \int_0^{\epsilon_F} d\epsilon N(\epsilon), \quad (8)$$

where the integrated DoS is

$$N(\epsilon) = -\frac{1}{\pi} \text{Im} \sum_{\epsilon_\mu} \ln(\epsilon_\mu - \epsilon_+) = -\frac{1}{\pi} \text{Im} \ln S(\epsilon_+), \quad (9)$$

where $\epsilon_+ = \epsilon + i0$, $S(\epsilon) = \prod_{\epsilon_\mu} b_\mu(\epsilon_\mu - \epsilon) = S_d(\gamma = -\epsilon)$, where ϵ_μ are the eigenvalues of the Schrödinger equation.

For a clean system, these eigenvalues are the same as those of the diffusion equation, with the substitutions $D \rightarrow \hbar/(2m)$ and $2e \rightarrow e$ [20].

Comparing Eqs. (8) and (9) with Eq. (7), we can now formally relate M_0 and the HF magnetization $\langle M_{ee} \rangle$ of the same diffusive system:

$$M_0 = -\lim_{\lambda_0 \rightarrow 0} \frac{1}{\lambda_0} \text{Im}[\langle M_{ee} \rangle(-\epsilon_F - i0)]. \quad (10)$$

This limit corresponds to taking the first-order contribution in λ_0 . As a simple illustration, consider the orbital magnetic susceptibility of an infinite disordered plane. For a disordered conductor, it is the Aslamasov-Larkin susceptibility χ_{AL} [18]:

$$\chi_{\text{AL}} = \frac{4}{3} \frac{\hbar D}{\phi_0^2} \ln \frac{\ln T_0 \tau \phi}{\ln T_0 \tau_e}. \quad (11)$$

$T_0 = \epsilon_F e^{1/\lambda_0}$ and $\phi_0 = h/e$ is the flux quantum. After replacing γ by $-\epsilon_F - i0$, taking the imaginary part of the logarithm, and replacing D and $2e$, we recover

the Landau susceptibility for the clean system: $\chi_0 = -e^2/24\pi m$.

We now calculate the spectral determinant for quasi-1D graphs. By solving the diffusion equation on each link, and then imposing Kirchoff-type conditions on the nodes of the graph, the problem is reduced to the solution of a system of N linear equations relating the eigenvalues at the N nodes. Let us introduce the $N \times N$ matrix M [21]:

$$M_{\alpha\alpha} = \sum_{\beta} \coth(\eta_{\alpha\beta}), \quad M_{\alpha\beta} = -\frac{e^{i\theta_{\alpha\beta}}}{\sinh \eta_{\alpha\beta}}. \quad (12)$$

The sum \sum_{β} extends to all the nodes β connected to the node α ; $l_{\alpha\beta}$ is the length of the link between α and β . $\eta_{\alpha\beta} = l_{\alpha\beta}/L_\phi$. The off-diagonal coefficient $M_{\alpha\beta}$ is nonzero only if there is a link connecting the nodes α and β . $\theta_{\alpha\beta} = (4\pi/\phi_0) \int_{\alpha}^{\beta} A dl$ is the circulation of the vector potential between α and β . The authors of Ref. [13] derived a relation between \mathcal{P} and the elements of the matrix M and its inverse $T = M^{-1}$:

$$\begin{aligned} 2\gamma \mathcal{P} &= (N - N_B) + \sum_{(\alpha\beta)} \eta_{\alpha\beta} F_{\alpha\beta} \\ F_{\alpha\beta} &= \coth \eta_{\alpha\beta} - \frac{(T_{\alpha\alpha} + T_{\beta\beta})}{\sinh^2 \eta_{\alpha\beta}} \\ &\quad + 2 \text{Re}(e^{i\theta_{\alpha\beta}} T_{\beta\alpha}) \frac{\cosh \eta_{\alpha\beta}}{\sinh^2 \eta_{\alpha\beta}}, \end{aligned} \quad (13)$$

where N_B is the number of links in the graph. Using the equality $\text{Tr}(M^{-1} \partial_\gamma M) = \partial_\gamma \ln \det M$ and recognizing in each term of (13) the partial derivative with respect to γ , we find that Eq. (13) can be rewritten as $\mathcal{P} = \frac{\partial}{\partial \gamma} \ln S_d$ where the spectral determinant S_d is given by

$$S_d = \left(\frac{L_\phi}{L_0} \right)^{N_B - N} \prod_{(\alpha\beta)} \sinh \eta_{\alpha\beta} \det M, \quad (14)$$

apart from a multiplicative factor independent of γ (or L_ϕ). L_0 is an arbitrary length. We have thus transformed the spectral determinant which is an infinite product in a finite product related to $\det M$.

As an example, we consider a disordered ring of perimeter L , to which one arm of length b is attached. The spectral determinant is equal to

$$S_d = \sinh R y \sinh y + 2 \cosh R y [\cosh y - \cos(4\pi\varphi)],$$

where $\varphi = \phi/\phi_0$ is the ratio between the flux ϕ threading the ring and the flux quantum. $y = (L/L_\phi)$ and $R = b/L$. Thus the average magnetization is

$$\begin{aligned} \langle M_{ee} \rangle &= \frac{\lambda_0 e D}{\pi^2} \int_{L/L_\phi}^{\infty} \\ &\quad \times \frac{2 \sin 4\pi\varphi y dy}{\tanh R y \sinh y + 2(\cosh y - \cos 4\pi\varphi)}. \end{aligned} \quad (15)$$

If there is no arm ($R = 0$), we retrieve the classical expression for the average magnetization of a disordered ring [22]. We notice that, in the limit $b \gg L_\phi$, the

magnetization remains finite and is equal to 2/3 of the single ring magnetization (for $L_\phi \lesssim L$, which corresponds to typical experimental values).

We want first to outline once more the connection between ballistic and disordered regimes. From Eq. (15) and with the mapping (10), $\gamma \rightarrow -E - i0$ and $L/L_\phi \rightarrow ikL$, where k is the wave vector of the solutions of the Schrödinger equation, we immediately recover the current in a one channel ballistic ring [23].

Let us come back to a diffusive network made of connected rings. Experimentally, the coherence length is of the order of the perimeter of one ring so that only a few harmonics of the flux dependence may be observed. It is then useful to make a perturbative expansion. We split the matrix as $M = D - N$, where D is a diagonal matrix: $D_{\alpha\alpha} = M_{\alpha\alpha} \approx z_\alpha$ to the lowest order in L_ϕ (z_α is the connectivity of the node α); $N_{\alpha\beta} = M_{\alpha\beta} \approx 2e^{-L_{\alpha\beta}/L_\phi} e^{i\theta_{\alpha\beta}}$. Expanding $\ln \det(I - D^{-1}N) = \text{Tr}[\ln(I - D^{-1}N)]$, we have

$$\ln \det M = \ln \det D - \sum_{n \geq 1} \frac{1}{n} \text{Tr}[(D^{-1}N)^n]. \quad (16)$$

We call "loop" l , a set of n nodes linked by n wires in a closed loop. The length L_l of a loop l is the sum of the lengths of the $n(l)$ links. The flux dependent part of $\ln S$ can be expanded as

$$\ln S = -2 \sum_{\{l\}} \frac{2}{z_1} \dots \frac{2}{z_{n(l)}} e^{-L_l/L_\phi} \cos(4\pi\phi_l/\phi_0). \quad (17)$$

ϕ_l is the flux enclosed by the loop l .

For example, we consider the cases shown on Fig. 1. Reducing the above sum to elementary loops l_0 (with two nodes), so that $n(l_0) = 2$, the first harmonics of the total magnetization, to the first order in λ_0 , is

$$\langle M_{ee} \rangle = 2G \frac{\lambda_0 e D}{\pi^2} (L/L_\phi + 1) e^{-L/L_\phi}, \quad (18)$$

where $G = \sum_{\{l_0\}} 4/(z_1 z_2)$. z_1 and z_2 are the connectivity of the two nodes of each loop. The sum is made over the m rings of the structure (see Fig. 1). In particular, it is $G = (m + 2)/4$ for an open necklace of m rings and $G = m/4$ for a closed necklace. The same reduction factors were obtained for weak-localization corrections after lengthy calculations for $m = 1, 2, 3, \infty$ in Ref. [13]. For the isolated ring, one recovers the known first harmonics [11] and the above reduction factor 2/3 for the ring with one arm. For a harmonic p of the

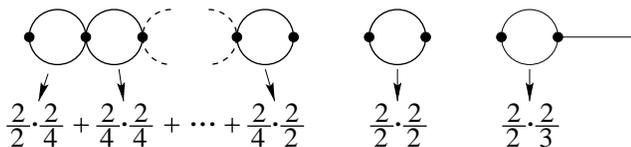


FIG. 1. Connectivity factors $(2/z_1)(2/z_2)$ entering in the loop expansion (17), for a series of identical connected rings, a single ring, and a ring with one arm.

magnetization, corresponding to a winding number p in the diffusion process, one should renormalize the interaction parameter because of the Cooper renormalization $\lambda = \lambda_0/[1 + \lambda_0 \ln \epsilon_F/(\pi E_c/p^2)]$ [5].

Figure 2 displays a comparison between the magnetization of different networks of connected rings, evaluated numerically using Eqs. (7) and (14). The perturbative expansions are in extremely good agreement with exact results as soon as the coherence length is smaller than the perimeter of one ring (see dashed lines in Fig. 2).

Finally, we calculate the distribution of the local current on each link of the graph. On a link $(\alpha\beta)$, the average current is given by the derivative of the Hartree-Fock energy correction E_{HF} to the vector potential $A(r)$, where r is any point belonging to the link $(\alpha\beta)$:

$$\langle J_{\alpha\beta}(r) \rangle = - \frac{\delta E_{\text{HF}}}{\delta A(r)} = \frac{\lambda}{\pi} \int_\gamma^{+\infty} d\gamma_1 \frac{\delta \ln S}{\delta A(r)}, \quad (19)$$

$$\frac{\delta \ln S}{\delta A(r)} = \text{Tr} \left(M^{-1} \frac{\delta}{\delta A(r)} M \right) = \frac{16\pi}{\phi_0} \text{Im}(M_{\beta\alpha}^{-1} M_{\alpha\beta}). \quad (20)$$

Fluctuations of the current corresponding to Eq. (6) can be obtained similarly [24].

In the limit $L_\phi \lesssim L$ considered above, the current distribution can also be derived quite simply. Indeed, in this approximation, the total magnetization can be written as a sum $\langle M_{ee} \rangle \sim \sum_k \langle m_k \rangle$, where $\langle m_k \rangle$ is identified as the magnetization of a plaquette k and depends on the position of this plaquette in the array. It is given by the rules of Eq. (17) and is shown on Fig. 3 for a regular square lattice. The average persistent current flowing in one link is the

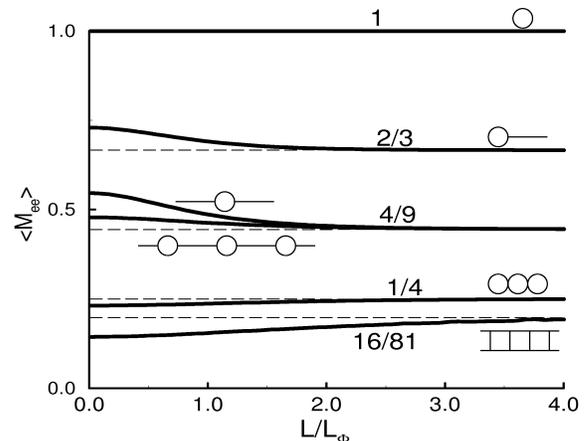


FIG. 2. Magnetization per ring for networks of connected rings normalized to the single ring magnetization, calculated exactly (solid lines) and with the loop expansion (dashed lines). The perimeter of all rings and sidearm lengths are equal to L . The three bottom curves correspond to regular networks made of an infinite number of rings (only three are represented). In these cases, the magnetization has been divided by the number of rings. The flux threading all rings is $\phi = \phi_0/8$.

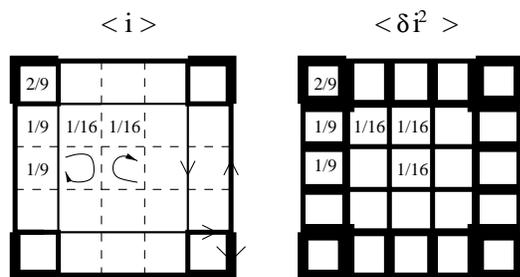


FIG. 3. Current distribution (left: average current; right: variance) for a square network, in the limit $L_\phi \leq L$, i.e., when the flux dependence of the current is harmonic. The numbers show the amplitude of the average and typical magnetization per plaquette, in units of the magnetization of the single ring. It is maximum at the corner plaquettes. The thickness of each link is proportional to the amplitude of the current on this link, obtained by difference (sum) of the average (typical squared) magnetizations of the plaquettes neighboring the link.

difference of the two plaquette currents neighboring it. The distribution of average current is sketched on Fig. 3.

The fluctuations can be described in the same way, namely, as a sum of terms which can be interpreted as fluctuations of the magnetization of one plaquette. Thus the fluctuations of plaquettes are independent, and the fluctuations of current in one link are the sum of the fluctuations of its two nearby plaquette currents.

In conclusion, we have developed a formalism which relates *directly* the persistent current, and the transport properties (although not detailed in this Letter) to the determinant of a matrix which describes the connectivity of the graph. From a loop expansion of this determinant, simple predictions for the magnetization and the spatial distribution of the persistent current in any geometry can now be compared with forthcoming experiments on connected and disconnected rings. We have also found a correspondence between the phase coherent contribution to the orbital magnetism of a disordered interacting system and the orbital response of the corresponding clean noninteracting system.

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