

Landau Diamagnetism and Magnetization of Interacting Disordered Conductors

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We show how the orbital magnetization of an interacting diffusive electron gas can be simply related to the magnetization of the noninteracting system having the same geometry. This result is applied to the persistent current of a mesoscopic ring and to the relation between Landau diamagnetism and the interaction correction to the magnetization of diffusive systems. The field dependence of this interaction contribution can be deduced directly from the de Haas–van Alphen oscillations of the free electron gas. Known results for the free orbital magnetism of finite systems can be used to derive the interaction contribution in the diffusive regime in various geometries.

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In recent years, there have been many theoretical works on the thermodynamic properties of mesoscopic electronic systems, in particular, concerning their orbital magnetism [1–3]. The simplest description of metals deals with noninteracting electrons in the absence of disorder. The correction to Landau susceptibility due to electron-electron interactions and phase coherence has been worked out by Altshuler *et al.* [2]. Similarly, the persistent current in mesoscopic rings has been extensively studied. The simplest description of this effect was first done in a strictly one-dimensional (1D) picture of noninteracting electrons [4] and the effect of diffusion and interaction was described later by Ambegaokar and Eckern [5] and Schmid [6].

The very simple approach for free electrons and the more sophisticated description of interacting electrons in a disordered potential have been developed in a completely independent way. Here, we show how these descriptions are closely related. The main result of this Letter is a simple relation between the response of a clean *noninteracting* electron gas and the response of a *diffusive* electron system *in the presence of interactions*. This result originates from the very similar structures of the Schrödinger equation and of the diffusion equation which describe the two systems.

As a first example, we show how the persistent current of a 1D ballistic ring is related to the current of a quasi-1D diffusive ring in the presence of interactions [7]. Then we show how the interaction contribution to the orbital magnetism of any diffusive system can be deduced immediately from the orbital response of the same noninteracting system. As a second example, we show how the interaction contribution to the susceptibility of a bulk diffusive system is derived directly from the Landau susceptibility. Then, from the de Haas–van Alphen oscillations of the free electron gas, we deduce the field dependence of the interaction induced magnetization. Finally, we use this mapping to derive the finite size corrections (in L_φ/L) in the diffusive case from the $1/k_F L$ corrections of the magnetization of the clean system.

Classically, the probability $p(\mathbf{r}, \mathbf{r}', \omega)$ for a particle to diffuse from a point \mathbf{r} to another point \mathbf{r}' is the solution of

the diffusion equation,

$$(-i\omega + \gamma - D\nabla_{\mathbf{r}'}^2)p_\gamma(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}'). \quad (1)$$

D is the diffusion coefficient. This probability has actually two parts, a purely classical one (the Diffuson) and an interference part (the Cooperon) which results from interference between time reversed trajectories. The Cooperon has to be taken at $\mathbf{r} = \mathbf{r}'$. In a magnetic field, it obeys Eq. (1), where ∇ has to be replaced by $\nabla + 2ie\mathbf{A}/\hbar c$, \mathbf{A} being the vector potential. The charge ($-2e$) accounts for the pairing of time reversed trajectories which are supposed to propagate coherently up to a time τ_ϕ . $\gamma = 1/\tau_\phi$ and $L_\varphi = \sqrt{D\tau_\phi}$ is the phase coherence length.

The probability $p_\gamma(\mathbf{r}, \mathbf{r}', \omega = 0)$ has the same structure as the disordered averaged (retarded) Green's function $\overline{G}_\epsilon^R(\mathbf{r}, \mathbf{r}')$ of the Schrödinger equation for a free particle of energy ϵ and charge $-e$ in a disordered potential:

$$\left(\epsilon + i\frac{\hbar}{2\tau_e} + \frac{\hbar^2}{2m}\nabla_{\mathbf{r}'}^2\right)\overline{G}_\epsilon^R(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (2)$$

where τ_e is the elastic mean-free path. In a field, $\nabla \rightarrow \nabla + ie\mathbf{A}/\hbar c$. The solutions of Eqs. (1) and (2) can be written as

$$p_\gamma(\mathbf{r}, \mathbf{r}') = p_\gamma(\mathbf{r}, \mathbf{r}', \omega = 0) = \sum_n \frac{\psi_n^*(\mathbf{r})\psi_n(\mathbf{r}')}{\gamma + E_n^d}, \quad (3)$$

and

$$\overline{G}_\epsilon^R(\mathbf{r}, \mathbf{r}') = \sum_n \frac{\psi_n^*(\mathbf{r})\psi_n(\mathbf{r}')}{\epsilon + i\frac{\hbar}{2\tau_e} - E_n^s}, \quad (4)$$

where the eigenvalues $E_n^{d,s}$ are the solutions of similar equations,

$$-D\Delta\psi_n = E_n^d\psi_n, \quad -\frac{\hbar^2}{2m}\Delta\psi_n = E_n^s\psi_n, \quad (5)$$

with the mapping from the diffusion to the Schrödinger

problem:

$$\begin{aligned} D &\rightarrow \frac{\hbar}{2m}, \\ 2e &\rightarrow e, \\ \hbar\gamma &\rightarrow -\epsilon - i\frac{\hbar}{2\tau_e}. \end{aligned} \quad (6)$$

It has long been recognized that a Diffuson (or a Cooperon) behaves similar to a free particle with an effective mass $m^* = \hbar/2D$. The goal of this Letter is to study the consequences of this mapping on the orbital magnetism of clean and diffusive systems.

For a disordered finite system of size L , the Thouless energy E_c , given by $\hbar D/L^2$, is equivalent to the mean interlevel spacing $\Delta = \hbar^2/2mL^2$ of the eigenvalues of the Schrödinger equation. More interesting is the relation deduced from Eq. (6):

$$\frac{L}{L_\varphi} \rightarrow -ik_F L - \frac{L}{2l_e}, \quad (7)$$

where $l_e = v_F\tau_e$. $1/\tau_e$ spreads the levels of the Schrödinger equation while $1/\tau_\varphi$ spreads those of the diffusion equation. Inelastic disorder on the Cooperon plays, thus, the same role as elastic disorder on a free particle. More important, the relation (7) expresses that the limit $k_F L \gg 1$ for the clean system corresponds to the macroscopic limit $L \gg L_\varphi$. Inversely, the mesoscopic limit $L \ll L_\varphi$ corresponds to having only one Schrödinger particle in a box ($k_F L \ll 1$).

Let us now apply this mapping to the calculation of the magnetization. First, the $T = 0$ K magnetic moment of the free electron gas (including spin) can be written as

$$M = \frac{\partial}{\partial B} \mathcal{N}(\epsilon_F, \mathcal{B}), \quad (8)$$

where $\mathcal{N}(\epsilon, \mathcal{B})$ is the double integral of the total density of states $\rho(\epsilon, B)$. This contribution is known as the Landau magnetization. Then taking into account electron-electron interactions in the Hartree-Fock picture gives an additional contribution [1]. For a completely screened interaction $U(\mathbf{r} - \mathbf{r}') = U\delta(\mathbf{r} - \mathbf{r}')$ [8], this contribution is given by

$$\begin{aligned} \langle M_{ee} \rangle &= -\frac{U}{4} \frac{\partial}{\partial B} \int \langle n^2(\mathbf{r}) \rangle d\mathbf{r} \\ &= -U \frac{\partial}{\partial B} \int \langle \rho(\mathbf{r}, \omega_1) \rho(\mathbf{r}, \omega_2) \rangle d\mathbf{r} d\omega_1 d\omega_2. \end{aligned} \quad (9)$$

This expression contains the Hartree and Fock contributions. $n(r)$ is the local density. $\rho(\mathbf{r}, \omega)$ is the local density of states (per spin direction). The average product $\langle \rho(\mathbf{r}, \omega_1) \rho(\mathbf{r}, \omega_2) \rangle$ is nothing but the Fourier transform $p_\gamma(\mathbf{r}, \mathbf{r}, \omega_1 - \omega_2)$ of the return probability $p_\gamma(\mathbf{r}, \mathbf{r}, t)$, so that one gets finally [10]

$$\langle M_{ee} \rangle = -\frac{\lambda_0 \hbar}{\pi} \frac{\partial}{\partial B} \int \frac{P_\gamma(t)}{t^2} dt, \quad (10)$$

where $P_\gamma(t) = \int p_\gamma(\mathbf{r}, \mathbf{r}, t) d\mathbf{r}$ in the space integrated return probability. $\lambda_0 = U\rho_0$ is a dimensionless interaction parameter, and ρ_0 is the average density of states (per spin direction). Writing the density of states as

$$\rho(\epsilon) = -\frac{1}{\pi} \int \text{Im} \bar{G}_\epsilon^R(\mathbf{r}, \mathbf{r}) d\mathbf{r}, \quad (11)$$

and the integrated return probability as

$$\int P_\gamma(t) dt = \int p_\gamma(\mathbf{r}, \mathbf{r}) d\mathbf{r}, \quad (12)$$

one obtains immediately from Eqs. (8) and (10) that the two magnetizations are related [since the $1/t^2$ term in Eq. (10) is equivalent to a double integral over γ]:

$$M \cong -\frac{1}{\lambda_0} \text{Im} \left[\langle M_{ee} \rangle \left(\gamma = -\frac{\epsilon_F}{\hbar} - i0 \right) \right]. \quad (13)$$

The sign \cong means that the two quantities are equal, provided the substitutions (6) have been made. It should then be remembered that Eq. (10) corresponds to taking the first order contribution in λ_0 to the grand potential. It is known that, taking into account higher diagrams in the Cooper channel, one has to renormalize the interaction parameter which becomes energy dependent $\lambda(\epsilon)$ [9,11,12]:

$$\lambda_0 \rightarrow \lambda(\epsilon) = \lambda_0 / \left(1 + \lambda_0 \ln \frac{\epsilon_F}{\epsilon} \right) = 1 / \ln \frac{T_0}{\epsilon}, \quad (14)$$

where T_0 is defined as $T_0 = \epsilon_F e^{1/\lambda_0}$. Then the relation (13) can be simply modified as

$$M = -\lim_{\lambda_0 \rightarrow 0} \frac{1}{\lambda_0} \text{Im} \left[\langle M_{ee} \rangle \left(\gamma = -\frac{\epsilon_F}{\hbar} - i0 \right) \right]. \quad (15)$$

As an example, we consider the case of a 1D diffusive ring of perimeter L pierced by a Aharonov-Bohm flux ϕ . Starting from the flux dependent part of the return probability,

$$P(t) = \frac{L}{4\pi Dt} \sum_{p=1}^{\infty} e^{-(p^2 L^2)/(4Dt)} \cos 4\pi p \varphi, \quad (16)$$

where $\varphi = \phi/\phi_0$, ϕ_0 being the flux quantum, one simply gets, from Eq. (10), the harmonic expansion of the average persistent current due to interactions:

$$\langle I_{ee} \rangle = 16\lambda_0 \frac{E_c}{\phi_0} \sum_{p=1}^{\infty} \frac{1}{p^2} \left(1 + p \frac{L}{L_\varphi} \right) e^{-pL/L_\varphi} \sin 4\pi p \varphi. \quad (17)$$

This result, for $L_\varphi = \infty$, was first obtained by Ambe-gaokar and Eckern (AE) [5]. It was then generalized to the case where L_φ is finite [10]. Using the relation (13), one deduces immediately the average persistent current for a clean 1D ring (clean means here that there is no diffusion. Disorder is taken into account only by a finite mean-free

path $l_e = v_F \tau_e$):

$$I = \frac{2}{\pi} I_0 \sum_{p=1}^{\infty} \frac{1}{p} \left(\cos p k_F L - \frac{\sin p k_F L}{p k_F L} \right) \times e^{-pL/2l_e} \sin 2\pi p \varphi, \quad (18)$$

with $I_0 = e v_F / L$. This result has first been obtained for the case $k_F L \gg 1$ (in this case, the $\sin x/x$ term cancels) in the absence of disorder ($l_e = \infty$) [4]. Note that the correspondence between the AE current and the current of the ballistic ring is not trivial. The leading term in $k_F L$ for the clean case originates from the leading term in L/L_φ in the diffusive case. Therefore, taking simply the AE result for the mesoscopic limit ($L_\varphi = \infty$) would not have produced the correct result for the clean ring. In other words, the $k_F L \gg 1$ limit corresponds to the macroscopic limit for the diffusive case. We will return to this point later, where we will show how to derive the L_φ/L corrections to diffusive magnetization from perimeter corrections in the ballistic case.

Deducing the magnetization of a clean system from the one of the interacting system may not appear as the most useful procedure. More interesting is deducing the properties of an interacting medium from those of the noninteracting one, i.e., to invert Eq. (13). This inversion is given by

$$\langle M_{ee} \rangle \cong -\frac{\lambda_0}{\pi} \int_0^\infty \frac{M(\epsilon)}{\epsilon + \hbar\gamma} d\epsilon, \quad (19)$$

with the substitution (6). Defining \tilde{M} as the magnetization of a free particle of mass $\hbar/2D$ and charge $2e$, so that $\tilde{M}(\epsilon) \cong M(\epsilon)$, one can rewrite

$$\langle M_{ee} \rangle = -\frac{\lambda_0}{\pi} \int_0^\infty \frac{\tilde{M}(\epsilon)}{\epsilon + \hbar\gamma} d\epsilon. \quad (20)$$

Again, recognizing that Cooper Channel renormalization modifies the interaction parameter, the energy dependence of this parameter can be incorporated exactly in the integral so that [13]

$$\langle M_{ee} \rangle = -\frac{1}{\pi} \int_0^\infty \lambda(\epsilon) \frac{\tilde{M}(\epsilon)}{\epsilon + \hbar\gamma} d\epsilon. \quad (21)$$

This is the main result of this paper. It gives straightforwardly *the magnetization of an interacting electron gas in terms of the magnetization of the same noninteracting system*.

As an example, we now consider the orbital response of a 2D clean system. The (spinless) Landau susceptibility gives the nonoscillating part of the orbital response. It is given by $\chi(\epsilon) = -e^2/(24\pi m)$ and is independent of the energy $\chi(\epsilon) = \chi_L$. Then, using the mapping (6), the susceptibility of the Cooperon is $\tilde{\chi}(\epsilon) = -\frac{4\pi}{3} \frac{\hbar D}{\phi_0^2} = -4\chi_L(\epsilon_F \tau_e)/\hbar$. From Eq. (21), one deduces the interaction part of the susceptibility [2,3]:

$$\chi_{ee} = \frac{4}{3} \frac{\hbar D}{\phi_0^2} \ln \frac{\ln T_0 \tau_\varphi / \hbar}{\ln T_0 \tau_e / \hbar} = 4 |\chi_L| \frac{\epsilon_F \tau_e}{\hbar} \ln \frac{\ln T_0 \tau_\varphi / \hbar}{\ln T_0 \tau_e / \hbar}. \quad (22)$$

An ultraviolet cutoff $1/\tau_e$ has been added in order to cure the divergence at large energy.

In 3D, the Landau susceptibility becomes energy dependent $\chi(\epsilon) = -e^2 k_F(\epsilon)/(24\pi^2 m) \propto \sqrt{\epsilon}$, so is the susceptibility $\tilde{\chi}(\epsilon)$ of the Cooperon, $\tilde{\chi}(\epsilon) = -8\chi_L \sqrt{\frac{\epsilon \tau_e}{3\hbar}}$. Contrary to the 2D case where the susceptibility was constant in energy and of order $\epsilon_F \tau_e$, integration in energy gives here a much smaller contribution. Using Eq. (21), one gets the interaction correction in 3D:

$$\frac{\chi_{ee}}{|\chi_L|} = \frac{16}{\pi\sqrt{3}} \frac{1}{\ln T_0 \tau_e / \hbar}. \quad (23)$$

Consider again the 2D clean case. In addition to the Landau contribution, the de Haas–van Alphen effect expresses the oscillatory behavior of the grand potential in $1/B$, with the fundamental period $1/B_0 = e\hbar/m\epsilon_F$. The grand potential is given by [14]

$$\delta \mathcal{A}(B) = -\frac{1}{2} \chi_L B^2 \left(1 + \frac{12}{\pi^2} \sum_{s=1}^{\infty} \frac{(-1)^s}{s^2} \cos \frac{2\pi s \epsilon_F}{\hbar \omega_c} \right), \quad (24)$$

and the magnetic moment at fixed Fermi energy is given by $M = -\partial \delta \mathcal{A} / \partial B$. Its dependence versus field has the well-known sawtoothed behavior. One may wonder how this behavior translates into the language of interacting diffusive electrons. To simplify, we restrict ourselves to the first order in λ_0 . Using the mapping [(19) and (20)], one deduces the interaction contribution to the magnetization, in units of $\lambda_0 \hbar D / \phi_0^2$:

$$\langle M_{ee}(B) \rangle = \frac{4}{3} B \ln \frac{\tau_\varphi}{\tau_e} + \frac{8}{\pi^2} \frac{\partial}{\partial B} B^2 \sum_{s=1}^{\infty} \frac{(-1)^s}{s^2} f\left(2\pi s \frac{B_\varphi}{B}\right), \quad (25)$$

where $f(x) = -\text{ci}(x) \cos(x) - \text{si}(x) \sin(x)$. The fundamental frequency B_0 has been transformed into the characteristic field $B_\varphi = \hbar/(4eD\tau_\varphi)$. Through the mapping [(19) and (20)], the de Haas–van Alphen oscillations have been transformed into a dull magnetic field dependence of the magnetization $\langle M_{ee}(B) \rangle$ shown on Fig. 1. The magnetic susceptibility $\chi_{ee} = \frac{\partial \langle M_{ee} \rangle}{\partial B}$ is shown on Fig. 2.

Alternatively, the magnetization (25) could have been obtained from Eq. (10), with the following expansion of the return probability in a constant field:

$$P(t) = \frac{B/\phi_0}{\sinh 4\pi B D t / \phi_0} = \frac{1}{4\pi D t} \left(1 + 2 \sum_1^\infty (-1)^s \frac{t^2}{t^2 + a^2 s^2} \right), \quad (26)$$

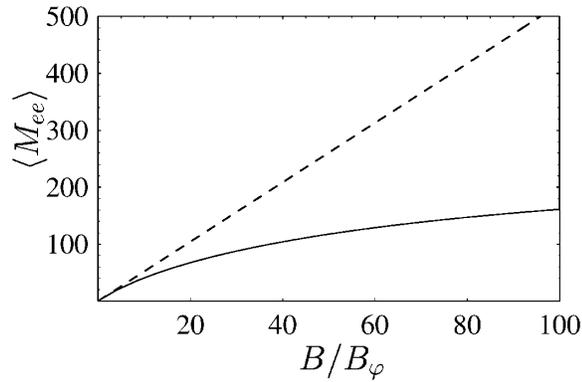


FIG. 1. Magnetization of a diffusive interacting electron gas calculated to first order in λ_0 , in units of $\lambda_0 \hbar D / \phi_0^2$. The dashed line shows the linear low field behavior [see Eq. (25)].

with $a = \phi_0 / (4BD)$. The result [(25)] can be easily generalized to all orders in λ_0 by considering the explicit dependence $\lambda(\epsilon)$ is Eq. (21).

Let us finally note that the limit $k_F L \gg 1$ for the Schrödinger equation corresponds to the macroscopic regime $L \gg L_\varphi$ for the diffusion equation. The opposite, so-called mesoscopic regime $L \ll L_\varphi$ would correspond to $k_F L \ll 1$, for which only the ground state is occupied. In the diffusive context, this ground state is called the *zero mode*. The crossover between the mesoscopic regime, where only a few modes are relevant to the macroscopic regime, where there is a quasicontinuum of diffusion modes, is quite difficult to describe [15]. It is then quite useful to know the finite size $1/k_F L$ corrections to the Landau susceptibility which have been extensively studied [16]. These corrections are usually of the form [16]

$$\chi(L) \approx \chi(\infty) \left(1 - \frac{\alpha}{k_F L} \right). \quad (27)$$

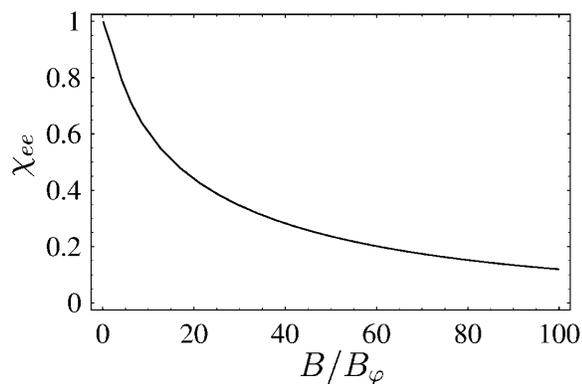


FIG. 2. Susceptibility of a diffusive interacting electron gas in units of $\lambda_0 \hbar D / \phi_0^2$. The amplitude at zero field is $4/3 \ln \tau_\varphi / \tau_e$ [see Eq. (25)].

Thus, knowing the finite size corrections to the Landau diamagnetism, one can get the L_φ/L corrections to the bulk susceptibility χ_{ee} . For $L \gg L_\varphi$, they are of the form [17]

$$\chi_{ee}(L) \approx \chi_{ee}(\infty) \left(1 - \alpha \frac{L_\varphi}{L} \right). \quad (28)$$

In conclusion, we have shown that the magnetization of a diffusive interacting electron gas can be deduced from the magnetization of the noninteracting system. This mapping allows the study of finite size properties of diffusive systems, in particular, the crossover between the macroscopic and the mesoscopic regimes.

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