Active Transmission Channels and Universal Conductance Fluctuations.

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Abstract. - The transport through a segment of a disordered system is determined by the eigenvalues of a large random matrix. The effectively independent active transmission channels are associated with these eigenvalues which are closest to unity. A decreasing number of those survives when the system's length increases. They determine the conductance and its fluctuations, which are found to be independent, within broad limits, of the size, disorder and nature of the system. This universality is due to the strong correlations in the spectra of large random matrices, providing a new insight on and generalizing the extremely interesting recent results of Altschuler, Lee and Stone.

1. Introduction.

An interesting concept which is clearly emerging in recent theoretical and experimental studies of quantum transport is that of non–self-averaging of the conductance of samples in the mesoscopic size range. This is the regime intermediate between microscopic and macroscopic where the sample's dimensions are less than a phase coherence length $L_{\varphi}$, which is the distance across which the electrons lose phase memory (typically by inelastic scattering). Since $L_{\varphi}$ can be of the order of a $\mu$m at $T \approx 1$ K, these effects are readily observable [2-5] in small wires in, say the $(0.1 \times 0.1 \times 1)$ $\mu$m$^3$ size range. Theoretically, if one considers an ensemble [6-11] of such systems made under identical macroscopic conditions, the conductance will fluctuate among members of the ensemble. Measuring the conductance, $g$, in units of $e^2/h$, ALTSCHULER, LEE and STONE [12] have found from a perturbative calculation for weak disorder the remarkable result that

$$ F \equiv \langle g^2 \rangle - \langle g \rangle^2 = \text{const} , $$

where the angular brackets signify ensemble averaging and the constant is a number of order unity, universal in the sense that it is independent of both disorder and sample's size, but it depends very weakly on sample shape, assuming somewhat different (but well defined!) values for e.g. squares, cubes and narrow thin (quasi 1D) wires [12].
This result is very different from the typical inverse square-of-the-size dependence of the relative fluctuation in classical inhomogeneous conduction. The physical reason for the quantum fluctuations is that, below $L_p$, interference effects are important and that the interference or speckle pattern is sensitive to the specific arrangement of the disorder in a given sample. This suggests that $g$ should be sensitive to a magnetic field, $B$ [14-16] or to changing the Fermi energy, $E_F$, as in a MOSFET device [17]. Here, we shall not dwell on the very important [7, 9, 11, 12] dependence on $B$ and $E_F$, except to note that the scales of these dependences are very easily understood physically in terms of the changes needed to modify the relevant phase differences across the sample by the order of $2\pi$. Thus, the scale of changes of $B$ is given by a flux change through the sample of the order of a flux quantum $\phi_0 = \hbar c/e$. Likewise, the scale of changes of $E_F$, $\Delta$, is given in the electronic diffusion regime by $\Delta = \hbar D/L^2$; $L$ being the sample's length. Thus, a measurement for a specific sample will produce sample-specific, reproducible [2] variation with a universal amplitude of the conductance as function of $B$ and/or $E_F$ on the above-mentioned scales. It should also be noted that such a measurement should be performed at temperatures low enough so that both $L_p \gg L$ and $k_B T \ll \hbar D/L^2$ (the latter is equivalent to $L \ll L_p = [\hbar D/k_B T]^2$). Qualitative remarks on the effects of higher temperatures will be given at the end of this letter. A particularly simple and clear case of such oscillations is that of a doubly connected sample, e.g. a small ring [1, 18-22], where the sample-specific dependence on an Aharonov-Bohm flux through its opening yields the recently observed [23, 24], $\phi_0$-periodic oscillations.

In fact, an incomplete estimate for these fluctuations has been given earlier in ref. [8], which considered the multichannel [24-27] Landauer [28] formula for the conductance. It was found that for a realistic ring, both $\phi_0$-periodic oscillation due to the flux in the hole and the aperiodic dependence on the flux in the ring's arms [2-5] were of the order of $1/N$ relative to the main, classical conductance. $N$ was the number of conduction channels assumed to be roughly equally effective and independent. It was stated in ref. [8] that the relevant $N$ might be less than the total number of channels (which is of the order of $N = k_F^2 A$, where $A$ is the wire's cross-section and $k_F$ the Fermi wave number). This may be due to channel correlations, strengthening of particular channels in analogy to a speckle effect [13], etc. So that the $1/N$ estimate is a lower bound.

It is the purpose of this letter to address the question [25, 29-31] of the determination of the effective independent channels and their number, $N_{eff}$, as a function of $N$ (the total channel number), $L$ and sample purity (parameterized by the elastic mean free path, $l$). We shall first present a qualitative argument which will yield

$$N_{eff} = N \frac{l}{L} \approx \langle g \rangle ,$$

which, with $\Delta g/g = N_{eff}^{-1}$, is immediately seen to be equivalent to (1). We shall then present a rigorous formulation [29, 30] of the problem, based on the theory of Pichard [31] and, making connection with the theory of random matrices [32-34], will establish (1) and (2) with one reasonable assumption, in the weak localization regime. Our results are, however, straightforwardly extendable to the strongly localized regime, which will also be briefly discussed. Comments on the effects of finite temperatures will be made in the final section.

2. A heuristic argument.

Following Lee and Stone [12] and the argument of Büttiker et al. [8], we shall use the simplified multichannel Landauer [24-27, 8] formula

$$g = 2 \text{tr} \, \hat{t} \hat{t}^+ ,$$

(3)
Fig. 1. – A schematic representation of a multichannel scatterer. The amplitude of the $N$ incoming channels from the left (right) are given by the components of the vector $a_l(a_r)$. Likewise, the amplitudes of the outgoing channels are $b_l(a_r)$. Where $t$ is the transmission matrix for the plane-wave amplitudes (i.e. for $b_r = 0$, $a_r = ta_l$ in fig. 1). Using (3) to approximate, the fuller conduction formula [24-27, 28] is valid for small enough transmission and/or large number of channels. The precise conditions for the approximate validity of eq. (3) have not been formulated in the literature. In fact, in some previous parallel addition multichannel formulae [24], these conditions were rather strict, since, e.g., a single channel with a strong transmission might have dominated $g$ and made (3) invalid. In the formula of ref. [8] and [25], which is the one we believe to be valid in the appropriate physical cases [18], one immediately sees that, at least if channel velocity factors are not important, the condition for the adequacy of eq. (3) is that $g << N$, which is approximately equivalent, as long as one is not in the strongly localized regime (in which, however, (3) is valid anyway) to

$$L \gg 1.$$  

We believe that the velocity factors should not alter this condition qualitatively, since, for example, the channels with very small velocities should have very small transmissions. Thus, (3) should be qualitatively correct (except for the asymmetries discussed in ref. [35, 36]) far away from the truly microscopic (or «ballistic») limit.

Now, the matrix $t$ is not multiplicative (i.e. for two consecutive segments, 1 and 2, $t_{12} \neq t_1 t_2$). On the other hand, the transfer matrix $T$, defined by $\begin{pmatrix} a_r \\ b_r \end{pmatrix} = T \begin{pmatrix} a_l \\ b_l \end{pmatrix}$, is multiplicative [29-31] and, thus, its eigenvalues do become $L$-th powers of the eigenvalues of some (Oseledec [37]) matrix for large enough $L$ (for convenience, we measure $L$ in atomic units). As found by Abrahams and Stephen [29], Azhel [25] and Pichard [31], $\text{tr} tt^+$ is related to a partial trace of the matrix $TT^+$, so that $g$ is essentially determined by multiplicative eigenvalues. All this will be presented and formalized in the next section. We believe, however, that it is useful to present the «bare bones» of the argument, which illustrates the physically most relevant ideas. Let us, thus, pretend that [25] $\text{tr} tt^+ = \sum x_j^2$, where $x_j$ are the real, positive and smaller-than-unity eigenvalues of some random matrix, arranged in descending magnitudes. For large $L$ only those close enough to unity will survive. In fact, the localization length, $\xi$, is defined as the $L$ for which only $x_0$ survives. Thus, for $L \geq \xi$

$$g = 2x_0^2 = 2 \exp \left[ -\frac{L}{\xi} \right],$$

where $\xi$ has to be defined so that $x_0 = \exp[1/\xi]$. There exists a certain $j = j_{\text{max}}$ beyond which the $x_j$'s decay quickly to zero. $j_{\text{max}}$ is defined by $x_{j_{\text{max}}}^2$ being smaller but of the order of unity, say $x_{j_{\text{max}}}^2 = 1/e$. We recall that the theory of random matrices [32-34], which will be appealed to for more quantitative discussion of the next section, finds that the eigenvalues' spacings tend to have relatively small fluctuations in spectral ranges where the average spacing does not change much (for this, one should take $j_{\text{max}} << N$, which will turn out to be equivalent to
Thus, \( x_{j_{\text{max}}} = \exp[-j_{\text{max}} \xi^{-1}] \) and \( L j_{\text{max}} \xi^{-1} = O(1) \). Since \( \xi \) is known to be of the order of \( NL \) [38], we find \( j_{\text{max}} = NUL \) (as a check we note that \( g \) itself is of the order of \( j_{\text{max}} \), as it should, for \( L < \xi \)). \( j_{\text{max}} \) plays the role of the effective number of active independent channels. In the eigenvector representation, only \( j_{\text{max}} \) such channels survive. In the plane-wave representation the transmissions are given by linear combinations of the independent surviving channels. (The others have been filtered out. The channel filtration has been mentioned in ref. [25] and [30].) This establishes that, as required below, \( N_{\text{eff}} \) is given by eq. (2). We note that \( N_{\text{eff}} = N \) for \( L = l \) and it decreases to unity for \( L = \xi \), where the system becomes eventually a single-channel or purely [29] a 1D one. The relative fluctuations in \( g \) are \( = 1/N_{\text{eff}} \), which reaches order unity around \( L = \xi \). This applies to both the aperiodic structure in wires and the \( h/e \)-periodic oscillation in rings, in agreement with simulations [11, 16] and experiment [23].


In the first part of this section we present for completeness a short derivation of a formula due to Pichard [31], according to which for small \( t \), where eq. (3) is valid,

\[
\text{tr} \, t t^+ = \text{tr} \frac{2}{T T^+ + (T T^+)^{-1} + 2I},
\]

(6)

\( I \) being the unit matrix. Recall that the \((2N \times 2N)\) matrix \( S \), defined by (see fig. 1)

\[
S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}, \qquad \begin{pmatrix} b_i \\ a_r \end{pmatrix} = S \begin{pmatrix} a_i \\ b_r \end{pmatrix},
\]

(7)

is unitary due to current conservation. \( T \) is not unitary (it is actually symplectic [30]). Writing \( T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \), the unitarity of \( S \) (current conservation) implies (using the well-known relationships between \( T \) and \( S \): \( T_1 = t - r' t^{-1} r', \ T_2 = r' t^{-1}, \ T_3 = -t^{-1} r, \ T_4 = t^{-1} \))

\[
T_4 T_4^* - T_3 T_3^* = T_1 T_1^* - T_2 T_2^* = I, \quad T_4 T_2^* - T_3 T_1^* = 0.
\]

(8)

In Pichard's original and more systematic treatment, (8) is obtained from the symplectic property of \( T \) which is also due to current conservation. It is helpful to consider the Hermitian (and symplectic) matrix \( T T^+ \). In fact, one finds from (8)

\[
T T^+ + (T T^+)^{-1} = 2 \begin{bmatrix} 2T_1 T_1^* - I & 0 \\ 0 & 2T_4 T_4^* - I \end{bmatrix}.
\]

(9)

Specializing to the case of small transmission, to be in the range validity of eq. (3), we obtain eq. (6), for \( \text{tr} \, t t^+ \), which is equal to \( \text{tr} \, t t^+ \).

We now define a matrix \( \Lambda \) by \( \Lambda^L = T T^+ \), with eigenvalues \( x_j \) denoted by \( \exp[\Delta_j] \), only those close to unity will be relevant. Due to the symplectic property the eigenvalues come in pairs \( \exp[\pm \Delta_j] \). Equation (6) yields

\[
g = 2 \sum \frac{1}{\cosh 2L \Delta_j + 1}.
\]

(10)
The matrix $A$ tends to a well-defined Oseledec [37] matrix as $L \gg \xi$. It is clear that the eigenvalues for which $L \Delta > 1$ are irrelevant. Since $g = j_{\text{max}} = N_{\text{eff}}$, the above estimate for $N_{\text{eff}}$ is valid. For a more precise treatment we note that for $L \leq \xi$, $A$ is a random matrix. The spectra of large random matrices are known to have some remarkable general properties [32-34]. It is far from obvious that our matrix $TT^*$ satisfies the conditions under which the above properties were proven, and this is even less clear for $\Delta$. However, these properties are believed [32, 33] to be valid for much wider classes of random matrices. Thus, we shall assume, and this is our only assumption, that the results of the theory of random matrices are applicable here. Since the basis reason for these properties is the eigenvalue repulsion, this assumption is reasonable for general features of the spectrum. The property of key relevance to us here is that of strong correlation, leading to a sort of long-range order, in the eigenvalues [34]. In fact, Dyson and Mehta [34] found that for any quantity, $W$, expressed in terms of a sum over some smooth function, $f$, of the eigenvalues $x_j$,

$$W = \sum_{j=0}^{n} f(x_j),$$

the variance of $W$ over the appropriate ensemble of random matrices is given by (see sect. 2 of ref. [34])

$$\langle W^2 \rangle - \langle W \rangle^2 = 2 \int_{-\infty}^{\infty} |t| \Phi(t) \Phi(-t) \, dt, \quad \Phi(t) = \int f(x) \exp[-2\pi i xt] \, dx.$$  

Very generally, this fluctuation (12) is shown from the results of Dyson and Mehta [34] to be independent of both $n$ and the level separation $D$ (which in our case is $O(1/\xi)$, as argued below). As Dyson and Mehta put it, $W$ can be measured with an error of order unity (rather than $O(N_{\text{eff}})$, as for an uncorrelated spectrum). This is the essence of the universality of the fluctuations. In our case the sum, eq. (10), is not cut off at a precise $n$. It does, however, converge strongly around a few times $j_{\text{max}}$ found above so that this should not affect the qualitative results. We use the matrix $A$ rather than its large power, $TT^*$, since the typical level separation of the former varies much less through the $= 2j_{\text{max}}$ levels closest to 1. To roughly estimate the numerical value for $\langle \Delta \varrho^2 \rangle$, we find from eqs. (6) and (10) that $f(x) = 2[x^L + x^{-L} + 1]^{-1} = 2(\cosh 2L \Delta + 1)^{-1}$, where $\Delta = \ln x = x - 1$. Since the relevant range for $\Delta$ is of order $1/L$, the linear approximation for $\Delta$ is reasonable. Using this $f(\Delta)$, we find from eq. (12) that $\langle \Delta \varrho^2 \rangle = 24 \zeta(3)/\pi^4 = 0.296$, independent of $L$ and $D$. Thus, while we have not obtained the exact number for $\langle \Delta \varrho^2 \rangle$ and its shape dependence, we did establish with some rigor that $\langle \Delta \varrho^2 \rangle$ is of order unity and independent of $L$, $N$ and $l$ and obtained a very rough estimate for it with very little effort. This derivation highlights the concept of the filtered effective transmission channels, which was argued above to be of substantial physical relevance. Moreover, while the derivation was presented in this section for the weakly localized regime, it is quantitatively extendable to the more general case of strong localization ($N_{\text{eff}} = 1$, $L \gg \xi$), as demonstrated in the next section. It should be possible to get information on the intermediate, few-channel, case as well.

4. The strongly localized, single-channel case.

In this case, only the two eigenvalues closest to unity survive in eq. (6). Since the eigenvalues of symplectic matrices come in inverse pairs, we denote the relevant pair as $x_0 = \exp[-1/2\xi]$ and $x_0^{-1}$, leading for $L \gg \xi$ to exponentially decaying $g$ (cf. eq. (5)). The
separation between $\Delta_0$ and $1/\Delta_0$ is $1/\xi$. Let us denote the average separation by $1/\xi_0$. Let us first consider the case $L = \xi_0$. The probability distribution for $1/\xi$ is known exactly in terms of the variable $t = \xi_0/\xi$ [34], where a very good approximation to it is the Wigner Surname [32, 34], $P_{W}(t) = (\pi/2) t \exp[-(\pi/4) t^2]$, from which it is straightforward to obtain all moments of $\ln g$ and all desired distributions. This is roughly valid for the Landauer-type conductance of an arbitrary wire once its length $L$ is comparable with $\xi_0$ and its temperature is so low that the conduction [39] is dominated by tunnelling through it. For $L \gg \xi$ it is clear that $\ln g$ and $1/\xi$ are [24] additive functions and the contributions of consecutive segments of size $\xi$ are independent [36]. Thus, the distribution of $\ln g$ sharpens up with increasing $L$ and becomes Gaussian for $L/\xi \to \infty$, when the Oseledec limit [30, 31] becomes valid. It should be possible to treat the whole range $L \geq \xi_0$ using these ideas.

5. Temperature effects, conclusions.

Before presenting our conclusions, we comment briefly on the effects of higher temperatures in the range $L_{\xi} \leq \xi$. The Landauer-type approach is also applicable, for $L \leq L_{\xi}$, at low temperatures, where the conductance $g(T)$ is given, as long as (3) is valid, by

$$g(T) = \int g(E) \left( \frac{\partial f}{\partial E} \right) dE$$

(13)

$f$ being the Fermi function and $g(E)$ is the zero temperature $g$ with $E = E_F$. For $L \geq L_{\xi}$, (13) can be used for every piece of the sample of size $L_{\xi}$ and the conductance of the whole sample is obtained by adding these conductances classically. One can consider two cases.

a) $L_{\xi} \ll L$, here the smearing of (13) is irrelevant on the scale of $L_{\xi}$ and

$$\frac{\langle \Delta g(T)^2 \rangle}{g(T)^2} = \frac{1}{\langle g(L_{\xi})^2 \rangle} \frac{L_{\xi}^2}{\text{Vol}}$$

where the last ratio is the number of «$L_{\xi}$ volumes» (segments of length $L_{\xi}$ for a wire thinner than $L_{\xi}$, squares of side $L_{\xi}$ for $2D$ thin films) in the sample's «volume», Vol.

b) $L_{\xi} \gg L$. Here the $\Delta g^2(0)$ of each $L_{\xi}$-volume experiences thermal averaging due to (13). This will give a further averaging factor of $\Delta_\xi/k_B T$ in $\langle \Delta g^2 \rangle$, where $\Delta_\xi = hD/L_{\xi}^2 = h/\tau_{\text{in}}$. A fuller calculation [40] shows that above two dimensions, this simple average is replaced by

$$\left( \frac{\Delta_\xi}{k_B T} \right)^{d-2} \left( \frac{h}{\tau_{\text{in}} k_B T} \right)^{d-2}$$

corresponding to classical addition of $L_T$-volumes. Finite temperatures and further interesting situations were very recently considered by ALTSCHULER and KHMELNITSKII [41].

The above thermal factors do reduce markedly the relative magnitude of the fluctuations. These fluctuations can still be observable, especially if they are made to be large. For $L_{\xi} = \xi$, the fluctuation on scale $L_{\xi}$ is of relative order unity, so that these fluctuations may be observable at low temperatures for macroscopic size samples.

To summarize: both the aperiodic fluctuations in a wire and the $h/e$-periodic Aharonov-Bohm oscillations in a ring are of relative size $1/N_{\text{eff}}$. $N_{\text{eff}}$, the number of active independent channels due to channel filtering, is given by (2), it varies from $-N$ on scale $l$ to $-1$ on scale $\xi$. 

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