Magneto-polarisability of mesoscopic rings

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Abstract. – We calculate the average polarisability of two-dimensional mesoscopic rings in the presence of an Aharonov-Bohm flux. The screening is taken into account self-consistently within a mean-field approximation. We investigate the effects of statistical ensemble, finite frequency and disorder. We emphasize geometrical effects which make the observation of field-dependent polarisability much more favourable on rings than on disks or spheres of comparable radius. The ratio of the flux-dependent to the flux-independent part is estimated for typical GaAs rings.

Transport properties of mesoscopic metallic conductors have been known for a long time to be quite sensitive to the quantum phase coherence of electronic wave functions at low temperature. More recently it has been shown both theoretically and experimentally [1] that the orbital magnetism also presents remarkable signatures of this phase coherence. The purpose of the present paper is to discuss to what extent the electrical polarisability, i.e. the response of a metallic sample to an electrostatic field, is also sensitive to mesoscopic effects. This quantity can be experimentally measured by inserting the mesoscopic samples into a capacitor. In the linear-response regime, the change in the capacitance is directly proportional to the average polarisability \( \alpha \) of the particles. If the typical size \( L \) of the particle is much larger than the Thomas-Fermi screening length \( \lambda_s \), the polarisability is mostly determined by the geometrical shape of the particle with a small correction of order \( \lambda_s/L \) [2]. If, furthermore, \( L \sim L_\phi \), the phase coherence length, one can expect to find a quantum correction \( \delta \alpha_Q \) to the polarisability: \( \alpha = \alpha_{TF} + \delta \alpha_Q \). The typical value of this magnetic-field–dependent quantum correction was estimated from diagrammatic theory [3] in disks and spheres. We have calculated the average of the magneto-polarisability of two-dimensional mesoscopic rings in the presence of an Aharonov-Bohm flux, with in-plane electric field. In this geometry \( \alpha_{TF} \) is of the order of \( R^3 \) and the quantum correction is of the order of \( \lambda_s/W \), where \( R \) and \( W \) are the radius and width of the ring. This correction is much larger than for disks or spheres of comparable radius with approximately the same value for \( \alpha = \alpha_{TF} \). We consider a time-dependent electric field which leads to two different limits according to whether the frequency is smaller or larger than the inverse inelastic time \( \gamma \). Nevertheless, both \( \omega \) and \( \gamma \)
are supposed to be small compared to the mean level spacing $\Delta$. We will consider the average value of the polarisability in grand canonical ensemble (GCE) as well as in the canonical ensemble (CE). In this letter we present mainly analytical results which have been confirmed by numerical simulations, to be published separately [4]. The agreement between our findings and the recent results of Efetov on disks and spheres [5] is only qualitative (in particular, we strongly disagree on the disorder dependence of the effect).

We have treated the screening of the applied electric field by the electrons within a mean-field approximation which leads to the one-electron Hamiltonian in an Aharonov-Bohm ring ($A$ is the static vector potential)

$$H(t) = \frac{(-i\nabla + eA)^2}{2m} + V_{\text{dis}}(r) + e\Phi_{\text{scr}}(r) \exp[\text{i}\omega t] + V_{\text{int}}(r),$$

where $\Phi_{\text{scr}}(r) = E_0 F(r)$ is the effective screened potential experienced by the electrons in the time-dependent external electric field $E_0 \exp[\text{i}\omega t]$ which is assumed to be small enough in order that linear response is valid. $V_{\text{dis}}$ is a disordered impurity potential and $V_{\text{int}}$ is the mean-field potential due to electron-electron interactions. In an Aharonov-Bohm geometry $\Phi_{\text{scr}}(r)$ and $V_{\text{int}}(r)$ depend on the magnetic flux, as pointed out by Büttiker [6]. In the present work we neglect $V_{\text{int}}(r)$ and consider single-electron eigenstates and eigenvalues, but we fully take into account the screening. $\Phi_{\text{scr}}(r)$ is related self-consistently to the field-induced shift of the electronic density $\delta n(r, E_0)$, according to

$$\begin{cases}
\Phi_{\text{scr}}(r) = \int \frac{e\delta n(r')}{4\pi e_0 |r' - r|} \text{d}r' + E_0 \cdot r, \\
\delta n(r) = \int \chi(r,r',\omega) \Phi_{\text{scr}}(r') \text{d}r'.
\end{cases}$$

The a.c. electric field induces a dipolar moment which is given by $\delta P_z(t) \equiv e \text{Tr}[(\rho(t) - \rho_0) \cdot X] \equiv \alpha E$, where $\text{Tr}$ denotes the trace over all states. $X$ is the projection of the position operator along the direction of the electric field and $\rho$ is the density matrix, solution of the master equation in the presence of the time-dependent perturbation:

$$i\frac{\partial \rho}{\partial t} = [H, \rho] - i\gamma (\rho - \rho_{\text{eq}}),$$

$\gamma$ is the inverse relaxation time towards equilibrium. $\rho_{\text{eq}}$ is the instantaneous equilibrium density matrix which satisfies $[H, \rho_{\text{eq}}] = 0$. Following the same steps as for the computation of the orbital susceptibility of Aharonov-Bohm rings, within linear-response theory [7], [8], the polarisability can be expressed as a function of eigenvalues $\epsilon_\alpha$ of the unperturbed Hamiltonian, matrix elements of the operators $X$ and $F(r)$ taken between the corresponding eigenstates, and populations $f_\alpha = \left( \text{exp} \left[ \frac{(\epsilon_\alpha - \mu)}{k_B T} \right] + 1 \right)^{-1}$,

$$\alpha = 2e^2 \left\{ \sum_{\alpha \neq \beta} f_\alpha - f_\beta \frac{\epsilon_\alpha - \epsilon_\beta - i\gamma}{\epsilon_\alpha - \epsilon_\beta + \omega - i\gamma} X_{\alpha,\beta} F(r, \phi)_{\beta,\alpha} + \frac{\gamma}{\gamma + i\omega} \sum_\alpha \frac{\partial f_\alpha}{\partial \epsilon_\alpha} X_{\alpha,\alpha} F(r, \phi)_{\alpha,\alpha} \right\}.$$

In the case where the dimensions of the sample are at least a few times larger than the screening length, $\lambda_s \approx \lambda_F$, it is reasonable to describe the screening by a Thomas-Fermi approximation [2], where the induced electronic density is proportional to the potential. For
a 2D system: \( \chi_{\text{TF}}(r, r') = (2\pi\epsilon_0/\lambda_0)\delta(r - r') \). The potential in the ring has then the form [4] \( F_{\text{TF}}(r) = f(r, \lambda, W) \cos \theta \), where the function \( f(r) \) depends on the aspect ratio of the ring. For \( W \ll R \) the charge density is nearly equally distributed between the 2 boundaries of the ring, on a length of the order \( \lambda_0 \), otherwise the charge is concentrated on the external boundary. The Thomas-Fermi polarisability is related to \( f(r) \) by \( \alpha_{\text{TF}} = (2\epsilon_0\pi^2/\lambda_0) \int f^2 f(r)dr \). For small \( \lambda_0 \), we recover the polarisability obtained by solving the Poisson equation, whose value depending on the ratio \( W/R \) extrapolates between the value of the 1D ring \( \alpha_{1D} = \epsilon_0\pi^2 R^3/\ln(R/W) \) and the value of the disk [9] \( \alpha_{2D} = \epsilon_0 \frac{16}{3}\pi R^3 \). As the ratio \( \lambda_0/W \) increases, size-dependent corrections decrease the value of \( \alpha_{\text{TF}} \).

In order to estimate the quantum correction to the polarisability, let us write the local electric susceptibility tensor as follows: \( \chi(r_1, r_2) = \chi_{\text{TF}}(r_1, r_2) + \delta \chi (r_1, r_2, \phi) \), where \( \delta \chi \) is the flux-dependent quantum correction. The same for the average screened potential: \( F(r, \phi) = F_{\text{TF}}(r) + \delta F(r, \phi) \). In first order of \( \delta \chi \) and \( \delta F \), which are self-consistently related through eq. (2), the quantum correction to the polarisability in the limit \( \omega, \gamma \ll \Delta \) can then be expressed as

\[
\delta \alpha(\phi) = 2\epsilon^2 \delta \left( \sum_{\alpha \neq \beta} \frac{f_{\alpha} - f_{\beta}}{\epsilon_{\alpha} - \epsilon_{\beta}} |F_{\alpha,\beta}^{\text{TF}}(r)|^2 + \frac{\gamma}{\gamma + i\omega} \sum_{\alpha} \frac{\partial f_{\alpha}}{\partial \epsilon_{\alpha}} |F_{\alpha,\alpha}^{\text{TF}}(r)|^2 \right).
\]

Note that this expression is very similar to the unscreened polarisability [10]-[12] where the matrix elements of the operator \( X \) are replaced by those of the average, flux-independent, screened potential. For the discussion of this quantity in the zero-frequency limit it will be important to distinguish between CE and GCE averages as well as between the order of the limits \( \gamma \gg \omega \rightarrow 0 \) (thermodynamic polarisability \( \alpha_{\text{TD}} \)) and \( \omega \gg \gamma \rightarrow 0 \) (dynamical polarisability \( \alpha_{\text{AD}} \)). On this point we disagree with the statement of Efetov who claims that all the limits should be identical [5]. In order to obtain the following results valid at zero temperature, we have used the expressions of the CE and GCE average population factors \( \langle f_{\alpha} - f_{\beta} \rangle \) and \( \langle \partial f_{\alpha}/\partial \epsilon_{\alpha} \rangle \) which have been extensively studied in [8], [13], [14] in the context of the ac magnetic response of Aharonov-Bohm rings. We call \( \delta \alpha \) the flux-dependent part of \( \alpha \).

In the GCE:

\[
\begin{align*}
\delta \alpha_{\text{GCE}}^{\text{GCE}} & = \frac{\epsilon^2}{E_F} \left( \sum_{\alpha \neq \beta} |F_{\alpha,\beta}^{\text{TF}}|^2 \right) = 0, \\
\delta \alpha_{\text{GCE}}^{\text{D}} & = \frac{2\epsilon^2}{E_F} \delta \left( \sum_{\alpha \neq \beta} |F_{\alpha,\beta}^{\text{TF}}|^2 \right) = -\frac{2\epsilon^2}{E_F} \delta \left( \sum_{\alpha} |F_{\alpha,\alpha}^{\text{TF}}|^2 \right),
\end{align*}
\]

where \( \Delta \) is the average level spacing and \( E_F \) is the Fermi energy. In the CE at zero temperature, the quantities \( \partial f_{\alpha}/\partial \epsilon_{\alpha} = 0 \), there is no contribution of diagonal matrix elements to the polarisability. The thermodynamic and dynamical polarisability are identical:

\[
\delta \alpha_{\text{TD}}^{\text{CE}} = \delta \alpha_{\text{D}}^{\text{CE}} = \frac{\epsilon^2}{E_F} \delta \left( \sum_{\alpha \neq \beta} |F_{\alpha,\beta}^{\text{TF}}|^2 \left( 1 + \frac{\Delta}{\epsilon_{\alpha} - \epsilon_{\beta}} \right) \right).
\]

Expression (6) shows that the flux-dependent polarisability in the GCE is directly related to the matrix elements of the screened potential \( F_{\text{TF}}^{\text{GCE}} \) which can be estimated from the Fourier spatial decomposition: \( F_{\text{TF}}^{\text{GCE}}(r) = \sum_{n=-M/2}^{M/2} F_n \exp[iq_n r] \), where \( q_n = \frac{n\pi}{W} \mathbf{u}_r + \frac{1}{R} \mathbf{u}_\theta \) and \( M \) is the number of transverse channels in the ring. Using the general relation established by
Aharonov-Bohm rings coupled to an electromagnetic resonator are sensitive enough to detect already noted in [17]. In order to estimate quantitatively the flux dependence of the screened electric field:

\[\langle |F_{\alpha,\beta}(r)|^2 \rangle = \frac{\Delta}{\pi E_c} \left( \frac{8R\lambda_s}{3\pi^2W} \right)^2; \]

\(E_c = h/\tau_D = hD/(2\pi R)^2\) is the Thouless energy and \(\tau_D\) is the diffusion time around the ring. This relation allows the generalization of the properties of the \(X\) operator established in [10], [16]. More interesting for our purpose, these matrix elements are also flux dependent. This flux dependence is \(\phi_0/2\) periodic and presents a minimum at \(\phi_0/4\), in contrast with the average square of the diagonal matrix element which is maximum at \(\phi_0/4\). The time-reversal symmetry property of the operator \(F^{TF}(r)\) implies that its diagonal matrix elements are real, even functions of flux and can be developed in successive powers of \(\cos(2\pi\phi/\phi_0)\). \(|F^{TF}_{\alpha,\alpha}|^2\) is maximum for multiple values of \(\phi_0/2\) and minimum for \(\phi_0/4[\phi_0/2]\). The opposite flux dependence for the non-diagonal matrix elements directly follows from the flux independence of the trace \(\sum_{\alpha,\beta} |F^{TF}_{\alpha,\beta}|^2 = \sum rF(r)\). Note that the flux dependences of the diagonal and non-diagonal matrix elements of the current operator, which is an odd function of flux, are just opposite as already noted in [17]. In order to estimate quantitatively the flux dependence of \(|F^{TF}_{\alpha,\alpha}|^2\), it is useful to relate the diagonal matrix element \(F^{TF}_{\alpha,\alpha}\) and the sensitivity of the eigenvalues \(\epsilon_\alpha\) to the screened electric field: \(F^{TF}_{\alpha,\alpha} = \partial\epsilon_\alpha/\partial E_0\). Since the electric field preserves time reversal symmetry [18], the typical derivative of the energy levels \(\langle \partial\epsilon_\alpha/\partial E_0 \rangle^2\) is proportional to \(1/\beta\), where the parameter \(\beta\) characterizes the symmetry class of the Hamiltonian, determined by another parameter (flux, for example) as a result

\[|F^{TF}_{\alpha,\alpha}|^2(\phi_0/4) \text{ (GUE ensemble)} = \frac{1}{2} |F^{TF}_{\alpha,\alpha}|^2(0) \text{ (GOE ensemble)}. \]

From eqs. (6), (9), (10) one deduces the magneto-polarisability in the GCE:

\[\frac{\delta\alpha}{\alpha} = \frac{\alpha^{GCE}_{\phi_0/4} - \alpha^{GCE}_{\phi_0/4}}{\alpha_0} = \frac{8}{3\pi^2} \frac{\Delta}{E_c} \frac{\lambda_s}{W} = \frac{8}{3\pi^2} \left( \frac{\Delta_T \lambda_s}{\hbar} \right) \frac{1}{W} ; \]

This result implies a positive low-field magneto-polarisability in agreement with findings of Efetov [5]. We also find an increase of the magneto-polarisability with disorder in the diffusive regime, in good agreement with numerical data depicted in fig. 1 but at odd with ref. [5], where a value of the order of \(\Delta \alpha/\alpha\) \(\tau_s \ll \tau_D\) is the elastic-scattering time found for \(\delta\alpha/\alpha\). A similar \(1/D\) dependence was also found for the typical value of the magneto-polarisability [3]. It is also clear from our results that the ring geometry is very favourable for the observation of quantum effects on the polarisability. The extrapolation for a disk or a sphere of same radius with comparable values of \(\alpha_{TF}\) yields \(\delta\alpha/\alpha = (\Delta/2E_c) \lambda_s/R\) for the disk with in-plane electric field and \(\delta\alpha/\alpha = (\Delta/2E_c) (\lambda_s/R)^2\) for the sphere. The relative correction is thus reduced by a factor \((W/R)^2\) for the 2D disk and a factor \(\lambda_s^3W^2/R^4\) for the sphere. We can estimate the relative magneto-polarisability of a ring made from a GaAs/GaAlAs heterojunction with the following parameters: \(L = 8 \mu m\), \(\lambda_s = 400 \AA\), \(l_s = 2 \mu m\), \(M = 10\), \(E_c = 7\Delta\), to be of the order of \(3 \cdot 10^{-3}\). Recent experiments measuring the a.c. complex conductance of such Aharonov-Bohm rings coupled to an electromagnetic resonator are sensitive enough to detect their magneto-polarisability [19].
Fig. 1. – Flux-dependent dynamical $\alpha_D$ and thermodynamical $\alpha_T$ polarisability in the GCE ensemble. These data are obtained from numerical simulations on the Anderson model for a ring $80 \times 8$ and 2 different values of disorder: $W_d = 1$ and $W_d = 2$ in the diffusive regime. The vertical axis has been arbitrarily shifted for the different curves.

If we now consider the canonical ensemble (CE), according to eq. (7), the flux-dependent correction to the polarisability in the CE at $T = 0$ depends both on matrix elements discussed above and energy levels. Assuming that energy levels and matrix elements are indeed independent random variables, it is possible to rewrite eq. (7):

$$\delta \alpha_{\text{CE}} = \delta \left[ \alpha_{\text{GCE}} + \frac{e^2}{\Delta} \langle (F^{TF})^2 \rangle \Delta \int \frac{R(\epsilon)}{\epsilon} d\epsilon \right],$$  \hspace{1cm} (12)$$

where $\langle (F^{TF})^2 \rangle \Delta$ is the average of $|F^{\alpha,\beta}|^2$ for $|\epsilon_\alpha - \epsilon_\beta| < \Delta$ and $R(\epsilon)$ is the probability of finding two levels separated by $\epsilon$ whose expression for the different symmetry classes is known from random matrix theory [20]. It is straightforward to show that

$$\int \left( \frac{R(\epsilon)^{\text{GUE}}}{\epsilon} - \frac{R(\epsilon)^{\text{GOE}}}{\epsilon} \right) d\epsilon = -\frac{1}{2\Delta}. \hspace{1cm} (13)$$

The flux dependence of the average matrix element $\delta \langle (F^{TF})^2 \rangle$ is of the order of $-\Delta/E_c$; as a result

$$\delta \alpha_{\text{CE}} \simeq \delta \alpha_{\text{GCE}} \frac{\Delta}{E_c}. \hspace{1cm} (14)$$

In the limit where $g = E_c/\Delta \gg 1$, the level repulsion contribution to the magneto-polarisability compensates the flux dependence of $\alpha_{\text{GCE}}$ in eq. (12). As a result the flux dependence of the polarisability in the CE is strongly reduced compared to the GCE at zero temperature. However it is not obvious whether this result obtained with single-electron wave functions is going to survive if electron-electron interactions are taken into account. These differences between CE and GCE averages are also expected to disappear at $T \geq E_c$.

We have estimated quantum corrections to the average electrical dynamical polarisability of 2D Aharonov-Bohm rings which give rise to a low-field positive magneto-polarisability. This effect has been shown to be directly related to the flux dependence of the matrix elements of
the screened potential in the rings. In the CE at low temperature the magneto-polarisability is sensitive to level spacing statistics whose contribution cancels the flux dependence due to matrix elements; it is strongly reduced compared to the GCE value. It is possible to evaluate the magneto-polarisability of a typical GaAs/GaAlAs ring to be of the order of $\delta\alpha/\alpha = 3\cdot10^{-3}$. This result is encouraging for experiments at least when the limit $\omega > \gamma$ is achieved.

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We